

GREEN'S FUNCTION ASYMPTOTICS AND SHARP POINT-WISE INTERPOLATION INEQUALITIES

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ABSTRACT. We propose a general method for finding sharp constants in the imbeddings of the Sobolev spaces $H^m(\mathcal{M})$, defined on a n -dimensional Riemann manifold \mathcal{M} into the space of bounded continuous functions, where $m > n/2$. The method is based on the analysis of the asymptotics with respect to the spectral parameter of the Green's function of the elliptic operator of order $2m$, the domain of the square root of which defines the norm of the corresponding Sobolev space. The cases of the n -dimensional torus \mathbb{T}^n and n -dimensional sphere \mathbb{S}^n are treated in detail, as well as some manifolds with boundary. In certain cases when \mathcal{M} is compact, multiplicative inequalities with remainder terms of various types are obtained. Inequalities with correction term for periodic functions imply an improvement for the well-known Carlson inequalities.

This paper is dedicated to the memory of Professor M.I. Vishik

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2000 *Mathematics Subject Classification.* 26D10, 46E35, 52A40.

Key words and phrases. Sobolev inequality, interpolation inequalities, Green's function, sharp constants, Carlson inequality.

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1. INTRODUCTION

In this paper we propose a general method for finding sharp constants in multiplicative inequalities of Sobolev–Gagliardo–Nirenberg type characterizing the imbedding of the Hilbert Sobolev space $H^m(\mathcal{M})$ into the space of bounded continuous functions $C(\mathcal{M})$. Here \mathcal{M} is an n -dimensional manifold and $m > n/2$. The inequalities are as follows:

$$(1.1) \quad \|u\|_{L^\infty(\mathcal{M})}^2 \leq K \|u\|_{H^l(\mathcal{M})}^{2\theta} \|u\|_{H^m(\mathcal{M})}^{2(1-\theta)},$$

where $-\infty < l < n/2 < m < \infty$, so that $0 < \theta = \frac{2m-n}{2(m-l)} < 1$.

Of course, multiplicative inequalities are known to hold in a much more general L_p setting (see, for instance, [3],[25]), when the three norms in (1.1) are the L_p , L_q and L_r -norms, $1 \leq p, q, r \leq \infty$, and θ is accordingly defined by scale invariance; in this sense we shall be dealing only with the $L_\infty - L_2 - L_2$ case in this paper. In the one-dimensional case, when $\mathcal{M} = \mathbb{R}$, \mathbb{R}_+ , or $(0, L)$ the corresponding interpolation inequalities are called the inequalities for derivatives. There exists a vast literature devoted to them, see, for instance, [1],[20] and the references therein. On the whole line, the general $L_p - L_q - L_r$ case of a function and its first-order derivative was completely settled in [21].

The sharp constant in inequality (1.1) on \mathbb{R} was found in [23], the more complicated case of the half-line \mathbb{R}_+ was solved in [10], the value of the constant in closed form was obtained in [15].

The sharp constant in inequality (1.1) for periodic functions with zero mean was found in [12]. In particular, for $l = 0$ it was shown there that

$$\|u\|_\infty^2 \leq K(m) \|u\|^{2\theta} \|u^{(m)}\|^{2(1-\theta)}, \quad u \in H_{\text{per}}^m(0, 2\pi), \quad \int_0^{2\pi} u(x) dx = 0,$$

where $m > 1/2$, $\theta = 1 - 1/(2m)$ and

$$K(m) = \left(2m\theta^\theta (1-\theta)^{1-\theta} \sin \pi\theta \right)^{-1}.$$

The constant (which is, in fact, the same as in the case of the whole line) is sharp and no extremal functions exist. We also observe that the first order inequality, namely,

$$(1.2) \quad \|u\|_\infty^2 \leq \|u\| \|u'\|, \quad u \in H_{\text{per}}^1(0, 2\pi), \quad \int_0^{2\pi} u(x) dx = 0,$$

was, in fact, proved much earlier in [11], as a proof of the Carlson inequality. (In the end of this section we discuss the connection of our results with the Carlson inequality in greater detail.)

Sharp constants in inequalities (1.1) on the sphere \mathbb{S}^n were found in [13].

A comprehensive analysis of sharp constants in (1.1) and in logarithmic Brezis–Gallouet inequalities on the n -dimensional torus \mathbb{T}^n has been done in [2], where inequalities with correction terms were obtained for the first time. For example, the following inequalities hold

$$(1.3) \quad \begin{aligned} \|u\|_\infty^2 &\leq \|u\| \|u'\| - \frac{1}{\pi} \|u\|^2, & u \in H_{\text{per}}^1(0, 2\pi), \quad \int_0^{2\pi} u(x) dx = 0, \\ \|u\|_\infty^2 &\leq \frac{\sqrt{2}}{\sqrt[4]{27}} \|u\|^{3/2} \|u''\|^{1/2} - \frac{2}{3\pi} \|u\|^2, & u \in H_{\text{per}}^2(0, 2\pi), \quad \int_0^{2\pi} u(x) dx = 0, \end{aligned}$$

where all constants are sharp and no extremal functions exist. These inequalities have been verified in [2] with an essential help of numerics. Purely analytic proof of them is given in the present paper as one of the applications, see Theorem 3.7.

In the first part of this paper we describe a general method for finding sharp constants in inequalities of the type (1.1) including the inequalities with correction terms. Namely, let A and B be self-adjoint non-negative elliptic differential operators on \mathcal{M} of order $2m$ and $2l$, respectively. (To describe the main ideas we may assume for a moment in this introduction that $l = 0$ and $B = Id$.) By the elliptic regularity the kernel of A is finite dimensional and consists of smooth (orthonormal) functions:

$$\ker A = \text{span}\{\varphi_1, \dots, \varphi_k\}.$$

We set $\bar{H}^m := H^m(\mathcal{M}) \cap \ker A^\perp$ and $\bar{H}^l := H^l(\mathcal{M}) \cap \ker A^\perp$. Then the square roots of A and B define the equivalent norms on $\bar{H}^m(\mathcal{M})$ and $\bar{H}^l(\mathcal{M})$:

$$\|u\|_A^2 := (Au, u) \sim \|u\|_{H^m(\mathcal{M})}^2, \quad \|u\|_B^2 := (Bu, u) \sim \|u\|_{H^l(\mathcal{M})}^2$$

for $u \in \bar{H}^m$ and $u \in \bar{H}^l$, respectively.

We fix an arbitrary point $\xi \in \mathcal{M}$ and consider the following analog of inequality (1.1):

$$(1.4) \quad |u(\xi)|^2 \leq K \|u\|_B^{2\theta} \|u\|_A^{2(1-\theta)}.$$

In particular, we are interested in the sharp constant $K = K(\xi)$ in this inequality. To study this problem, we introduce the following maximization problem: for every number $D \geq \lambda_0$, where λ_0 is the first eigenvalue of $B^{-1/2}AB^{-1/2}$ in $\bar{H} := L_2(\mathcal{M}) \cap \ker A^\perp$, find $\mathbb{V}(\xi, D)$, solving

$$(1.5) \quad \mathbb{V}(\xi, D) := \sup \left\{ |u(\xi)|^2 : u \in \bar{H}^m, \quad \|u\|_B^2 = 1, \quad \|u\|_A^2 = D \right\}.$$

Then, the inequality

$$|u(\xi)|^2 \leq \|u\|_B^2 \mathbb{V} \left(\xi, \frac{\|u\|_A^2}{\|u\|_B^2} \right)$$

holds and, by definition, \mathbb{V} is the smallest function for which it holds. Thus, in particular, if the maximization problem (1.5) is solved, finding the best constant K in (1.4) is reduced to finding the smallest K for which the following inequality holds:

$$\mathbb{V}(\xi, D) \leq KD^{1-\theta}, \quad D \geq \lambda_0.$$

The solution of the maximization problem (1.5) can be expressed in terms of the Green's function of the following elliptic operator of order $2m$:

$$\mathbb{A}(\lambda) := A + \lambda B, \quad \lambda > -\lambda_0.$$

Namely, let $G_\lambda(x, \xi)$ be the Green's function of it in \bar{H} :

$$\mathbb{A}(\lambda)G_\lambda(\cdot, \xi) = \bar{\delta}(\cdot, \xi), \quad \bar{\delta}(x, \xi) := \delta(x, \xi) - \sum_{j=1}^k \varphi_j(x)\varphi_j(\xi),$$

where $\delta(x, \xi)$ is the Dirac delta function. From elliptic regularity we see that $G_\lambda(\cdot, \xi) \in \bar{H}^m \subset C(\mathcal{M})$. Then, as shown in Theorem 2.3, there exist a unique extremal function in (1.5)

$$u_{D,\xi}(x) = \frac{G_{\lambda(D)}(x, \xi)}{\|G_{\lambda(D)}(\cdot, \xi)\|_B}, \quad \text{so that} \quad \mathbb{V}(\xi, D) = \frac{G_{\lambda(D)}(\xi, \xi)^2}{\|G_{\lambda(D)}(\cdot, \xi)\|_B^2},$$

where $\lambda(D)$ is a monotone increasing function with $\lim_{D \rightarrow \infty} \lambda(D) = \infty$, $\lim_{D \rightarrow \lambda_0} \lambda(D) = -\lambda_0$, defined as a *unique* solution of

$$\|u_{D,\xi}\|_A^2 = D.$$

Moreover, as shown in Theorem 2.5, the sharp constant K in the multiplicative inequality (1.4) can be expressed in terms of the Green's function as follows:

$$K = K(\xi) := \frac{1}{\theta^\theta(1-\theta)^{1-\theta}} \cdot \sup_{\lambda > 0} \left\{ \lambda^\theta G_\lambda(\xi, \xi) \right\} < \infty.$$

In addition, the extremal function u_* exists if and only if the supremum with respect to λ is attained at a finite point $\lambda_* < \infty$ and then $u_*(x) = \text{const } G_{\lambda_*}(x, \xi)$. We note that the approach used in [12], [13] in the case of \mathbb{S}^n , $n \geq 1$ eventually reduces to the same one-dimensional maximization problem.

The further progress (inequalities with correction terms) is based on the knowledge of the asymptotic behavior of the Green's function as $\lambda \rightarrow \infty$:

$$(1.6) \quad G_\lambda(\xi, \xi) = \lambda^{-1} \left(g_1 \lambda^{1-\theta} + g_2 + g_3 \lambda^{\theta-1} + o(\lambda^{\theta-1}) \right),$$

where $g_1 = g_1(\xi) > 0$, $g_2 = g_2(\xi)$, $g_3 = g_3(\xi)$ are some given numbers. This expansion is assumed, and its verification is one of the main technical tasks in particular examples in the second part of the paper.

Given (1.7), we prove in Proposition 2.6 that the solution of (1.5) has the following asymptotics as $D \rightarrow \infty$:

$$(1.7) \quad \mathbb{V}(\xi, D) = g_1 S D^{1-\theta} + g_2 \frac{1}{\theta} - \frac{1}{2} S^{-1} \frac{g_2^2(1-\theta) - 2\theta g_1 g_3}{\theta^3 g_1} D^{\theta-1} + o(D^{\theta-1}), \quad S := \frac{1}{\theta^\theta (1-\theta)^{1-\theta}}.$$

Suppose that the third term is negative (this is always the case when $g_3 \leq 0$), then

$$(1.8) \quad \mathbb{V}(\xi, D) < g_1 S D^{1-\theta} + \frac{g_2}{\theta},$$

for large D . We shall see in certain examples in the second part of the paper that this inequality holds for all D . This implies the multiplicative inequality with correction term (which is negative if $g_2 < 0$):

$$|u(\xi)|^2 \leq g_1 S \|u\|_B^\theta \|u\|_A^{1-\theta} + \frac{g_2}{\theta} \|u\|_B^2$$

with best possible constants.

In the end of first part of the paper we give in Theorem 2.8 the following variational characterization of $\mathbb{V}(\xi, D)$:

$$\mathbb{V}(\xi, D) = \inf_{\lambda \in [-\lambda_0, \infty)} \{(\lambda + D) G_\lambda(\xi, \xi)\}.$$

This formula is very useful for proving in certain cases (which are few!) that (1.8) holds for all D . The scheme is as follows. Usually it is impossible to find the unique minimizer $\lambda(D)$ explicitly, but we always have the asymptotic formula for it

$$\lambda(D) = \frac{\theta}{1-\theta} D + \frac{g_2}{g_1} S D^\theta + \dots,$$

see (2.44). We somehow truncate this expansion and denote the result as $\lambda_*(D)$. Then to prove (1.8) we proceed as follows

$$\mathbb{V}(\xi, D) - g_1 S D^{1-\theta} - \frac{g_2}{\theta} \leq (\lambda_*(D) + D) G_{\lambda_*(D)}(\xi, \xi) - g_1 S D^{1-\theta} - \frac{g_2}{\theta}.$$

Now for a fixed ξ the right-hand side is an explicit function of D only and more or less standard estimates can be used to prove that it is negative for all D . In fact, inequalities (1.3) as well as a number of other inequalities with lower order correctors mentioned below are verified using this scheme.

In the second part of the paper we consider examples and applications of the general approach described in the first part and we use the above scheme in Theorem 3.7 for purely analytic proof of inequalities (1.3) (the proof in [2] involves some reliable computer calculations).

We first deal with manifolds without boundary and consider the case $A = (-\Delta)^m$, $B = (-\Delta)^l$, where Δ is the Laplace–Beltrami operator. Then $\bar{H}^s(\mathcal{M}) = H^s(\mathcal{M}) \cap \{\int_{\mathcal{M}} u(x) d\mathcal{M} = 0\}$. The asymptotic formula (1.6) for the Green's function on the torus \mathbb{T}^n is obtained by the Poisson summation formula. This works when $l = 0$. For $l > 0$ there is a singularity, which can be removed by differentiation. This, in turn, produces the problem of finding integration constants.

The corresponding technique was proposed in [2] and it is further developed here. We consider only one example $n = 3$, $m = 2$, $l = 1$ and prove the following inequality on the three-dimensional torus \mathbb{T}^3 :

$$\|u\|_\infty^2 \leq \frac{1}{2\pi} \|\nabla u\| \|\Delta u\| - \frac{-\beta_3}{4\pi^3} \|\nabla u\|^2, \quad u \in \bar{H}^2(\mathbb{T}^3),$$

where the integration constant β_3 is expressed in terms of a super-exponentially convergent series, $\beta_3 = -8.91363291758515127$. Both constants are sharp and no extremals exist.

Next we study inequalities on spheres. On \mathbb{S}^2 we consider the case when $A = (-\Delta)^m$, $m > 1/2$, and $B = I$. The corresponding Green's function is independent of ξ and is given by the series

$$G(\lambda) = \frac{1}{4\pi} \mu^m \sum_{n=1}^{\infty} (2n+1) \varphi(\mu n(n+1)), \quad \text{where } \mu = \lambda^{-1/m}, \quad \varphi(x) = \frac{1}{x^m + 1}.$$

Thus, we need to find the asymptotic behavior of functions of the type

$$F(\mu) = \sum_{n=1}^{\infty} (2n+1) f(\mu n(n+1)) \quad \text{as } \mu \rightarrow 0,$$

where f is sufficiently smooth and sufficiently fast decays at infinity. This is achieved with the help of the Euler–Maclaurin formula in Lemma 3.14:

$$F(\mu) = \frac{1}{\mu} \int_0^\infty f(x) dx - \frac{2}{3} f(0) - \frac{1}{15} \mu f'(0) + O(\mu^2).$$

This gives the asymptotic expansion of the type (1.6) for the Green's function and, hence, the asymptotic expansion (1.7) for the solution $\mathbb{V}(D)$ of the corresponding maximization problem, in which the second and the third terms turn out to be both negative. Therefore a negative correction term may exist. For $m = 2$ we show that this is indeed the case and the following inequality (with two sharp constants and no extremal functions) holds for $u \in \bar{H}^2(\mathbb{S}^2)$:

$$\|u\|_\infty^2 \leq \frac{1}{4} \|u\| \|\Delta u\| - \frac{1}{3\pi} \|\nabla u\|^2.$$

For larger m ($m = 3$, $m = 4$) the negative correction terms still exist, but are smaller than the second terms in the expansion for $\mathbb{V}(D)$.

On \mathbb{S}^3 we consider only one example with $A = (-\Delta)^2$ and $B = -\Delta$. The series expressing the Green's function can be summed in closed form, and along the same lines we obtain a sharp inequality for $u \in \bar{H}^2(\mathbb{S}^3)$:

$$\|u\|_\infty^2 \leq \frac{1}{2\pi} \|\nabla u\| \|\Delta u\| - \frac{3}{4\pi^2} \|\nabla u\|^2.$$

In the remaining part of the paper we consider manifolds with boundary. We first prove a sharp multiplicative inequality on the half-line for the Bessel operator [17].

Then we consider the case when $u \in H_0^1(0, L)$. The correction term still exists, but is exponentially small, namely, the following inequality holds:

$$\|u\|_\infty^2 \leq \|u\| \|u'\| \left(1 - 2e^{-\frac{L\|u'\|}{\|u\|}}\right).$$

Both coefficients on the right-hand side are sharp and no extremal functions exist.

For a second order inequality on the interval $(0, L)$

$$\|u\|_\infty^2 \leq K \|u\|_{L_2(0,L)}^{3/2} \|u''\|_{L_2(0,L)}^{1/2},$$

in the case when $u \in H_0^2(0, L)$ we can use extension by zero, and therefore K is the same as on \mathbb{R} , namely, $K = \frac{\sqrt{2}}{\sqrt[4]{27}}$ (see, [23], and also (3.2)). In going over to a wider space $u \in H_0^1(0, L) \cap H^2(0, L)$ the constant may increase. In Theorem 3.21 we show that this is indeed

the case: $K = \frac{\sqrt[4]{2}}{\sqrt[4]{27}} \cdot \coth \frac{\pi}{2} = \frac{\sqrt[4]{2}}{\sqrt[4]{27}} \cdot 1.09033 \dots$, and, in addition, there exists a unique extremal function.

A somewhat opposite result is obtained in Theorem 3.23: the constant $\frac{1}{2\pi}$ in the inequality

$$\|u\|_\infty^2 \leq \frac{1}{2\pi} \|\nabla u\| \|\Delta u\|,$$

is sharp both for $u \in H_0^2(\Omega)$ and $u \in H_0^1(\Omega) \cap H^2(\Omega)$, $\Omega \subset \mathbb{R}^3$.

In conclusion we observe that inequalities of the type (1.1) have applications to various problems in partial differential equations and mathematical physics. For example, the far-going generalization to the matrix-valued case [6] of the simplest inequality

$$\|u\|^2 \leq \|u\| \|u'\|, \quad u \in H^1(\mathbb{R})$$

gives the best-known estimates of the Lieb–Thirring constants for the negative trace of the Schrödinger operators in \mathbb{R}^n [19].

Accordingly, inequalities with correction terms (1.3) imply by the method of [7] a simultaneous bound for the negative trace and the number of negative eigenvalues for the Schrödinger operators on \mathbb{S}^1 [14].

Finally, inequalities (1.3) provide an improvement to the well-known Carlson inequality [5] (see also [18] and the references therein): for $a_k \geq 0$

$$(1.9) \quad \left(\sum_{k=1}^{\infty} a_k \right)^2 \leq \pi \left(\sum_{k=1}^{\infty} a_k^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} k^2 a_k^2 \right)^{1/2},$$

where the constant π is sharp and the strict inequality holds unless all $a_k = 0$. In fact, as shown in [11], inequalities (1.2) and (1.9) are equivalent, hence an improvement to (1.2) results in an improvement to (1.9).

Given a sequence $a_k \geq 0$, we set $a_0 = 0$ and consider a real-valued function

$$u(x) = \sum_{k=-\infty}^{\infty} a_{|k|} e^{ikx}$$

with mean value zero, for which

$$\|u\|_\infty = u(0) = 2 \sum_{k=1}^{\infty} a_k, \quad \|u\|^2 = 4\pi \sum_{k=1}^{\infty} a_k^2, \quad \|u'\|^2 = 4\pi \sum_{k=1}^{\infty} k^2 a_k^2.$$

Now inequalities (1.3) can be reformulated as follows

$$(1.10) \quad \begin{aligned} \left(\sum_{k=1}^{\infty} a_k \right)^2 &\leq \pi \left(\sum_{k=1}^{\infty} a_k^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} k^2 a_k^2 \right)^{1/2} - \sum_{k=1}^{\infty} a_k^2, \\ \left(\sum_{k=1}^{\infty} a_k \right)^2 &\leq \frac{\sqrt{2}\pi}{\sqrt[4]{27}} \left(\sum_{k=1}^{\infty} a_k^2 \right)^{3/4} \left(\sum_{k=1}^{\infty} k^4 a_k^2 \right)^{1/4} - \frac{2}{3} \sum_{k=1}^{\infty} a_k^2, \end{aligned}$$

where all constants are sharp and the inequalities are strict unless all $a_k = 0$. Moreover, the interpolation inequalities on the tori \mathbb{T}^n discussed above can be naturally considered as multi-dimensional generalizations of the Carlson inequality.

In this paper we use standard notation. Sometime the C -norm (or L_∞ -norm) is denoted by $\|\cdot\|_\infty$ and the L_2 -norm is denoted by $\|\cdot\|$.

2. PART I. GENERAL THEORY

2.1. Assumptions and preliminaries. Let \mathcal{M} be an n -dimensional compact Riemann manifold and let

$$(u, v) := \int_{\mathcal{M}} u(x)v(x) \text{mes}(dx)$$

be the standard scalar product in $H := L^2(\mathcal{M}, \text{mes})$ (where mes stands for the measure on \mathcal{M} associated with the Riemann metric). As usual, we define the Sobolev spaces $W^{l,p}(\mathcal{M})$, $1 \leq p \leq \infty$ as spaces of distributions whose derivatives up to order l belong to $L^p(\mathcal{M})$ (this definition works for integer l only, for non-integer and/or negative l , the spaces $W^{l,p}(\mathcal{M})$ are defined in a standard way using the interpolation and duality methods, see e.g., [25] for the details). For the case $p = 2$, we will denote by $H^m = H^m(\mathcal{M})$ the Sobolev space $W^{l,2}(\mathcal{M})$.

We assume that A is an elliptic self-adjoint differential operator of order $2m$ on \mathcal{M} with smooth coefficients. This operator is supposed to be non-negative

$$(2.1) \quad (Au, u) \geq 0, \quad u \in H^m,$$

although it may have a non-zero kernel. Then, due to the classical elliptic theory (see e.g., [25]), the kernel is finite-dimensional and is generated by smooth functions

$$(2.2) \quad \ker A := \text{span}\{\varphi_1, \dots, \varphi_k\}, \quad \varphi_i \in C^\infty(\mathcal{M}), \quad A\varphi_i = 0.$$

Without loss of generality, we may assume that

$$\|\varphi_i\|_H = 1 \quad \text{and} \quad (\varphi_i, \varphi_j) = 0, \quad i \neq j.$$

We denote by \bar{H} the orthogonal complement of $\ker A$ in $H = L^2(\mathcal{M})$ and define

$$(2.3) \quad \bar{H}^s := H^s(\mathcal{M}) \cap \bar{H}, \quad s \in \mathbb{R}.$$

Then, due to the elliptic theory, A is an isomorphism between \bar{H}^{s+2m} and \bar{H}^s for all $s \in \mathbb{R}$ and, in particular, the equivalent norm in \bar{H}^m is given by

$$(2.4) \quad \|u\|_A^2 := (Au, u), \quad u \in \bar{H}^m.$$

We also introduce the second elliptic non-negative and self-adjoint differential operator B of order $2l < 2m$ on \mathcal{M} with smooth coefficients such that

$$(2.5) \quad \ker B \subset \ker A, \quad \text{and} \quad B \ker A \subset \ker A.$$

Then, as not difficult to see, the operator B is an isomorphism between \bar{H}^{s+2l} and \bar{H}^s for all $s \in \mathbb{R}$ and, in particular, the equivalent norm in \bar{H}^l is given by

$$(2.6) \quad \|u\|_B^2 := (Bu, u), \quad u \in \bar{H}^l.$$

In addition, we have the analogue of the Poincare inequality:

$$(2.7) \quad \|u\|_B^2 \leq \lambda_0^{-1} \|u\|_A^2, \quad u \in \bar{H}^m,$$

where $\lambda_0 > 0$ is the minimal eigenvalue of $B^{-1/2}AB^{-1/2}$ in \bar{H} .

2.2. Interpolation inequality and associated variational problem. If

$$(2.8) \quad l < \frac{n}{2} < m,$$

then for every $u \in \bar{H}^m$ the following interpolation inequality holds:

$$(2.9) \quad \|u\|_{C(\mathcal{M})} \leq C \|u\|_B^\theta \|u\|_A^{1-\theta}, \quad \frac{n}{2} = l\theta + m(1-\theta),$$

see [3], [25]. Our aim is to refine inequality (2.9) and, in particular, to find the best constant $C = C(n, m, l, \mathcal{M})$ or/and the lower order extra terms in it, etc. To this end, we fix an arbitrary point $\xi \in \mathcal{M}$ and a positive D and consider the following maximization problem:

$$(2.10) \quad \mathbb{V}(\xi, D) := \sup \left\{ |u(\xi)|^2 : u \in \bar{H}^m, \|u\|_B^2 = 1, \|u\|_A^2 = D \right\}.$$

Indeed, in view of (2.9) and (2.7) the function \mathbb{V} is well-defined for all $\xi \in \mathcal{M}$ and all $D \geq \lambda_0$. On the other hand, due to the homogeneity,

$$(2.11) \quad |u(\xi)|^2 \leq \|u\|_B^2 \mathbb{V} \left(\xi, \frac{\|u\|_A^2}{\|u\|_B^2} \right), \quad \xi \in \mathcal{M}, \quad u \in \bar{H}^m.$$

In particular, the best constant C in (2.9) is the minimal one for which the inequality

$$(2.12) \quad \sup_{\xi \in \mathbb{M}} \mathbb{V}(\xi, D) \leq C^2 D^{1-\theta}$$

holds for all $D \geq \lambda_0$. This reduces the study of inequality (2.9) to the investigation of the maximization problem (2.10).

2.3. Green's functions and reproducing functionals. In this subsection we prepare some technical tools which are necessary to give the analytic description of the function \mathbb{V} in terms of the Green's functions of the appropriate elliptic operators on \mathcal{M} . Namely, let

$$(2.13) \quad \mathbb{A}(\lambda) := A + \lambda B, \quad \lambda > -\lambda_0,$$

and let the function $G_\lambda(x, \xi)$ solve

$$(2.14) \quad \mathbb{A}(\lambda) G_\lambda(\cdot, \xi) = \bar{\delta}(\cdot, \xi), \quad \bar{\delta}(x, \xi) = \delta(x, \xi) - \sum_{i=1}^k \varphi_i(\xi) \varphi_i(x),$$

where $\delta(x, \xi)$ is the Dirac δ -function at point $\xi \in \mathcal{M}$ and $\bar{\delta}$ is its 'projection' on the space \bar{H} . Then, due to the Sobolev embedding theorem and (2.8),

$$\bar{\delta}(\cdot, \xi) \in \bar{H}^{-m}$$

and, therefore, due to the elliptic regularity,

$$(2.15) \quad G_\lambda(\cdot, \xi) \in \bar{H}^m \subset C(\mathcal{M}).$$

In addition, since $\mathbb{A}(\lambda)$ is self-adjoint, $G_\lambda(x, \xi) = G_\lambda(\xi, x)$ and, consequently, (2.15) implies that

$$G_\lambda \in C(\mathcal{M} \times \mathcal{M}).$$

Moreover,

$$(2.16) \quad u(\xi) = (\bar{\delta}(\cdot, \xi), u) = (\mathbb{A}(\lambda) G_\lambda(\cdot, \xi), u) = (G_\lambda(\cdot, \xi), \mathbb{A}(\lambda) u).$$

for all $u \in \bar{H}^m$. In particular, taking $u(x) := G_\lambda(x, \xi)$, we have

$$(2.17) \quad G_\lambda(\xi, \xi) = (\mathbb{A}(\lambda) G_\lambda(\cdot, \xi), G_\lambda(\cdot, \xi)) = \|G_\lambda(\cdot, \xi)\|_A^2 + \lambda \|G_\lambda(\cdot, \xi)\|_B^2 > 0.$$

The following simple lemma is nevertheless the main technical tool for the method of reproducing functionals and will allow us to find the analytic expression for the function \mathbb{V} .

Lemma 2.1. *Let the above assumptions hold. Then, for every $u \in \bar{H}^m$ and for every $\lambda > -\lambda_0$*

$$(2.18) \quad |u(\xi)|^2 \leq G_\lambda(\xi, \xi) (\|u\|_A^2 + \lambda \|u\|_B^2).$$

Moreover, the equality holds if and only if $u(x) = c G_\lambda(x, \xi)$ for some $c \in \mathbb{R}$.

Proof. Indeed, since $\mathbb{A}(\lambda)$ is positive definite, by the Cauchy-Schwartz inequality,

$$|u(\xi)|^2 = (G_\lambda(\cdot, \xi), \mathbb{A}(\lambda) u)^2 \leq (\mathbb{A}(\lambda) G_\lambda, G_\lambda) (\mathbb{A}(\lambda) u, u) = G_\lambda(\xi, \xi) (\|u\|_A^2 + \lambda \|u\|_B^2)$$

and the equality here holds if and only if $u(x) = c G_\lambda(x, \xi)$. Thus, the lemma is proved. \square

We note that, up to the moment, $\lambda \geq -\lambda_0$ is a free parameter in (2.18). The next lemma shows that this parameter can be chosen in such way that the quotient $\|G_\lambda(\cdot)\|_A^2/\|G_\lambda(\cdot, \xi)\|_B^2$ achieves any prescribed value.

Lemma 2.2. *The function*

$$(2.19) \quad D(\lambda) = D_\xi(\lambda) := \frac{\|G_\lambda(\cdot, \xi)\|_A^2}{\|G_\lambda(\cdot, \xi)\|_B^2}$$

is strictly increasing on $[-\lambda_0, \infty)$. Moreover,

$$(2.20) \quad \lim_{\lambda \rightarrow -\lambda_0} D(\lambda) = \lambda_0 \quad \text{and} \quad \lim_{\lambda \rightarrow +\infty} D(\lambda) = \infty,$$

so the inverse function $D \rightarrow \lambda(D)$ is well defined on $[\lambda_0, \infty)$.

Proof. We will show that $D'(\lambda) > 0$. To this end, we note that the function $G_\lambda(x, \xi)$ is smooth with respect to λ and the derivative $G'_\lambda(x, \xi) := \partial_\lambda G_\lambda(x, \xi)$ solves the equation

$$(2.21) \quad AG'_\lambda(\cdot, \xi) + \lambda BG'_\lambda(\cdot, \xi) = \mathbb{A}(\lambda)G'_\lambda(\cdot, \xi) = -BG_\lambda(\cdot, \xi),$$

so that

$$AG'_\lambda(\cdot, \xi) = -BG_\lambda(\cdot, \xi) - \lambda BG'_\lambda(\cdot, \xi), \quad G'_\lambda(\cdot, \xi) = -\mathbb{A}(\lambda)^{-1}BG_\lambda.$$

Therefore,

$$(2.22) \quad \begin{aligned} \frac{1}{2}\|G_\lambda(\cdot, \xi)\|_B^4 D'(\lambda) &= \\ &= \|G_\lambda(\cdot, \xi)\|_B^2 (AG'_\lambda(\cdot, \xi), G_\lambda(\cdot, \xi)) - (BG'_\lambda(\cdot, \xi), G_\lambda(\cdot, \xi))\|G_\lambda(\cdot, \xi)\|_A^2 = \\ &= -\|G_\lambda(\cdot, \xi)\|_B^4 - \lambda\|G_\lambda(\cdot, \xi)\|_B^2 (BG'_\lambda(\cdot, \xi), G_\lambda(\cdot, \xi)) - (BG'_\lambda(\cdot, \xi), G_\lambda(\cdot, \xi))\|G_\lambda(\cdot, \xi)\|_A^2 = \\ &= -\|G_\lambda(\cdot, \xi)\|_B^4 - (BG'_\lambda(\cdot, \xi), G_\lambda(\cdot, \xi))(\mathbb{A}(\lambda)G_\lambda(\cdot, \xi), G_\lambda(\cdot, \xi)) = \\ &= (\mathbb{A}^{-1}(\lambda)BG_\lambda(\cdot, \xi), BG_\lambda(\cdot, \xi))(\mathbb{A}(\lambda)G_\lambda(\cdot, \xi), G_\lambda(\cdot, \xi)) - \|G_\lambda(\cdot, \xi)\|_B^4. \end{aligned}$$

By the Cauchy-Schwartz inequality we have

$$(2.23) \quad \|v\|_B^4 = (Bv, v)^2 = (\mathbb{A}^{-1/2}(\lambda)Bv, \mathbb{A}^{1/2}(\lambda)v)^2 \leq \\ \leq \|\mathbb{A}^{-1/2}(\lambda)Bv\|_H^2 \|\mathbb{A}^{1/2}(\lambda)v\|_H^2 = (\mathbb{A}^{-1}(\lambda)Bv, Bv)(\mathbb{A}(\lambda)v, v)$$

for all $v \in \bar{H}^m$. Taking $v(x) = G_\lambda(x, \xi)$, we see from (2.22) that $D'(\lambda) \geq 0$. Moreover, equality in (2.23) holds only if

$$\alpha \mathbb{A}(\lambda)v = \beta Bv, \quad \alpha, \beta \in \mathbb{R}$$

and from elliptic regularity v should be at least C^∞ -smooth. However, $v = G_\lambda(\cdot, \xi)$ cannot be C^∞ -smooth, so the equality is impossible and $D'(\lambda) > 0$. Thus, we have proved that $D(\lambda)$ is strictly increasing.

We now need to verify (2.20). We start with the limit $\lambda \rightarrow -\lambda_0$. To solve (2.14) near $\lambda = -\lambda_0$, we introduce $w(x, \xi) = B^{1/2}G_\lambda(x, \xi)$. Then

$$(2.24) \quad B^{-1/2}AB^{-1/2}w + \lambda w = B^{-1/2}\bar{\delta}(x, \xi)$$

and, by definition, λ_0 is the smallest eigenvalue of the self-adjoint positive operator $B^{-1/2}AB^{-1/2}$ in \bar{H} . Clearly, this operator has compact inverse, so its spectrum is discrete. Let $\{\psi_k\}_{k=1}^\infty$ be family of the orthonormal eigenfunctions of it, and let the first p ($p \geq 1$) of them correspond to the smallest eigenvalue λ_0 . Clearly, $\psi_i \in C^\infty(\mathbb{M})$. Seeking w in the form

$$(2.25) \quad w(x, \xi) = \sum_{i=1}^p a_i \psi_i(x) + \sum_{i=p+1}^\infty a_i \psi_i(x) =: w_0 + w^\perp,$$

substituting this into (2.24), and taking the scalar product with ψ_1, \dots, ψ_p , we find the coefficients of the w_0 -part of the solution:

$$a_i = \frac{1}{\lambda + \lambda_0} (B^{-1/2} \bar{\delta}(\cdot, \xi), \psi_i) = \frac{\tilde{\psi}_i(\xi)}{\lambda + \lambda_0}, \quad i = 1, \dots, p,$$

where we set $\tilde{\psi}_i := B^{-1/2} \psi_i$. Observe that the w^\perp -part of the solution remains bounded as $\lambda \rightarrow -\lambda_0$. Applying $B^{-1/2}$ to (2.25) we find $G_\lambda(x, \xi)$:

$$(2.26) \quad G_\lambda(x, \xi) = \frac{1}{\lambda + \lambda_0} \sum_{i=1}^p \tilde{\psi}_i(\xi) \tilde{\psi}_i(x) + G_\lambda^\perp(x, \xi),$$

where the part $G_\lambda^\perp(x, \xi)$ remains bounded as $\lambda \rightarrow -\lambda_0$. Thus,

$$\lim_{\lambda \rightarrow -\lambda_0} D(\lambda) = \frac{\|\sum_{i=1}^p \tilde{\psi}_i(\xi) \tilde{\psi}_i(\cdot)\|_A^2}{\|\sum_{i=1}^p \tilde{\psi}_i(\xi) \tilde{\psi}_i(\cdot)\|_B^2} = \lambda_0 \frac{\sum_{i=1}^p \tilde{\psi}_i(\xi)^2}{\sum_{i=1}^p \tilde{\psi}_i(\xi)^2} = \lambda_0.$$

Let us consider the case $\lambda \rightarrow \infty$. Assume that (2.20) is wrong and we have

$$\lim_{\lambda \rightarrow \infty} D(\lambda) = D_{max} < \infty.$$

Then,

$$(2.27) \quad \|G_\lambda(\cdot, \xi)\|_A^2 \leq D_{max} \|G_\lambda(\cdot, \xi)\|_B^2.$$

Multiplying equation (2.14) by $G_\lambda(x, \xi)$ integrating in $x \in \mathcal{M}$, and using the embedding $\bar{H}^m \subset C$ and (2.27), we get

$$\|G_\lambda(\cdot, \xi)\|_A^2 + \lambda \|G_\lambda(\cdot, \xi)\|_B^2 = G_\lambda(\xi, \xi) \leq \|G_\lambda(\cdot, \xi)\|_C \leq C \|G_\lambda(\cdot, \xi)\|_A \leq C_1 \|G_\lambda(\cdot, \xi)\|_B.$$

Due to this estimate, we have

$$\|G_\lambda(\cdot, \xi)\|_A \leq C \|G_\lambda(\cdot, \xi)\|_B \leq C_2 \lambda^{-1},$$

where C_2 is independent of $\lambda \rightarrow \infty$. Thus, $w_\lambda(x, \xi) := \lambda G_\lambda(x, \xi)$ is uniformly bounded in \bar{H}^m and, without loss of generality, we may assume that $w_\lambda(\cdot, \xi) \rightarrow w_\infty(\cdot, \xi)$ weakly in this space. Then, obviously, $w_\infty(\cdot, \xi) \in \bar{H}^m$ and

$$Bw_\infty(\cdot, \xi) = \bar{\delta}(\cdot, \xi).$$

In particular, since $w_\infty \in \bar{H}^m$, we have $\bar{\delta}(\cdot, \xi) \in \bar{H}^{m-2l}$. Due to the assumption (2.8) we have $m - 2l > -n/2$, so that $\bar{\delta}(\cdot, \xi) \in H^{-s}$ for some $s < n/2$ which is impossible. Thus, (2.20) is proved and the lemma is also proved. \square

2.4. Main result. The aim of this subsection is to give the analytic expression for the function \mathbb{V} in terms of the Green's functions introduced above. This result is stated in the following theorem.

Theorem 2.3. *Let the above assumptions hold. Then, for every $D \in [\lambda_0, \infty)$ and every $\xi \in \mathcal{M}$, the supremum in (2.10) is the maximum and this maximum is achieved in a unique point*

$$(2.28) \quad u_{D, \xi}(x) := \frac{G_{\lambda(D)}(x, \xi)}{\|G_{\lambda(D)}(\cdot, \xi)\|_B},$$

where the function $\lambda(D)$ is defined in Lemma 2.2. In particular,

$$(2.29) \quad \mathbb{V}(\xi, D) = \left(\frac{G_{\lambda(D)}(\xi, \xi)}{\|G_{\lambda(D)}(\cdot, \xi)\|_B} \right)^2.$$

Proof. Indeed, let $u \in \bar{H}^m$, be such that $\|u\|_B = 1$ and $\|u\|_A^2 = D$. Then, according to Lemma 2.2, there is a unique $\lambda = \lambda(D)$ which solves (2.19). According to (2.18) with $\lambda = \lambda(D)$,

$$|u(\xi)|^2 \leq G_{\lambda(D)}(\xi, \xi)(D + \lambda(D))$$

and the equality here holds if and only if $u(x) = cG_{\lambda(D)}(x, \xi)$. Taking the B -norm of both sides of this equality, we see that $c = \|G_{\lambda(D)}(\cdot, \xi)\|_B^{-1}$ and $u(x) = u_{D, \xi}(x)$. This finishes the proof of the theorem. \square

Thus, in order to find \mathbb{V} , we need three functions

$$(2.30) \quad f_\xi(\lambda) := G_\lambda(\xi, \xi), \quad g_\xi(\lambda) := \|G_\lambda(\cdot, \xi)\|_B^2, \quad h_\xi(\lambda) := \|G_\lambda(\cdot, \xi)\|_A^2.$$

Then

$$(2.31) \quad D(\lambda) = \frac{h_\xi(\lambda)}{g_\xi(\lambda)} \quad \text{and} \quad \mathbb{V}(\xi, \lambda) := \mathbb{V}(\xi, D(\lambda)) = \frac{f_\xi(\lambda)^2}{g_\xi(\lambda)}, \quad \lambda \in [-\lambda_0, \infty)$$

and we have the parametric representation of the function $D \rightarrow \mathbb{V}(\xi, D)$. The next lemma shows that the functions g and h can be expressed in terms of f .

Lemma 2.4. *The functions f , g and h satisfy the following equalities*

$$(2.32) \quad f'_\xi(\lambda) := \frac{d}{d\lambda} f_\xi(\lambda) = -g_\xi(\lambda), \quad h_\xi(\lambda) = f_\xi(\lambda) + \lambda f'_\xi(\lambda).$$

Proof. Indeed, multiplying (2.14) by $G_\lambda(x, \xi)$ and integrating it over $x \in \mathcal{M}$, we have

$$(2.33) \quad f_\xi(\lambda) = G_\lambda(\xi, \xi) = \|G_\lambda(\cdot, \xi)\|_A^2 + \lambda \|G_\lambda(\cdot, \xi)\|_B^2 = h_\xi(\lambda) + \lambda g_\xi(\lambda).$$

Differentiating this formula with respect to λ and using (2.21), we get

$$(2.34) \quad f'_\xi(\lambda) = 2(\mathbb{A}(\lambda)G'_\lambda(\cdot, \xi), G_\lambda(\cdot, \xi)) + g_\xi(\lambda) = \\ = -2(BG_\lambda(\cdot, \xi), G_\lambda(\cdot, \xi)) + g_\xi(\lambda) = -2g_\xi(\lambda) + g_\xi(\lambda) = -g_\xi(\lambda).$$

It only remains to note that (2.34) and (2.33) imply (2.32) and finish the proof of the lemma. \square

The next result shows that the sharp constant in the inequality

$$(2.35) \quad |u(\xi)|^2 \leq K \|u\|_B^{2\theta} \|u\|_A^{2(1-\theta)}, \quad u \in \bar{H}^m.$$

is expressed in terms of the following *scalar* maximization problem again involving the Green's function $G_\lambda(\xi, \xi)$.

Theorem 2.5. *For a fixed $\xi \in \mathcal{M}$ the constant K in (2.35) is given by*

$$(2.36) \quad K = K(\xi) := \frac{1}{\theta^\theta (1-\theta)^{1-\theta}} \cdot \sup_{\lambda > 0} \left\{ \lambda^\theta G_\lambda(\xi, \xi) \right\} < \infty,$$

where θ is defined in (2.9) and the constant (2.36) is sharp. Furthermore, the extremal function in (2.35) exists if and only if the supremum in (2.36) is attained at a finite point λ_* .

Proof. We first show that (2.36) is finite. Indeed, using (2.17) and the inequality (2.9) (with the non-optimal constant $C!$), we have

$$\lambda^\theta G_\lambda(\xi, \xi)^2 \leq \lambda^\theta \|G_\lambda(\xi, \cdot)\|_{L^\infty}^2 \leq C(\lambda \|G_\lambda(\xi, \cdot)\|_B^2)^\theta (\|G_\lambda(\xi, \cdot)\|_A^2)^{1-\theta} \leq \\ \leq C(\|G_\lambda(\xi, \cdot)\|_A^2 + \lambda \|G_\lambda(\xi, \cdot)\|_B^2) = CG_\lambda(\xi, \xi)$$

and (2.36) is finite.

Let us check that (2.35) holds with $K = K(\xi)$. Indeed, let $u \in \bar{H}^m$ be arbitrary and let $\lambda := \frac{\theta}{1-\theta} \frac{\|u\|_A^2}{\|u\|_B^2}$. Then, using (2.18), we have

$$\begin{aligned}
 (2.37) \quad |u(\xi)|^2 &\leq G_\lambda(\xi, \xi) \|u\|_B^2 \left(\frac{\|u\|_A^2}{\|u\|_B^2} + \lambda \right) = \frac{1}{\theta} G_\lambda(\xi, \xi) \lambda \|u\|_B^2 = \frac{1}{\theta} G_\lambda(\xi, \xi) \lambda^\theta \lambda^{1-\theta} \|u\|_B^2 = \\
 &= \frac{1}{\theta} \lambda^\theta G_\lambda(\xi, \xi) \left(\frac{\theta}{1-\theta} \frac{\|u\|_A^2}{\|u\|_B^2} \right)^{1-\theta} \|u\|_B^2 = \frac{1}{\theta^\theta (1-\theta)^{1-\theta}} \cdot \lambda^\theta G_\lambda(\xi, \xi) \|u\|_B^{2\theta} \|u\|_A^{2(1-\theta)} \leq \\
 &\leq \frac{1}{\theta^\theta (1-\theta)^{1-\theta}} \cdot \sup_{\lambda > 0} \left\{ \lambda^\theta G_\lambda(\xi, \xi) \right\} \|u\|_B^{2\theta} \|u\|_A^{2(1-\theta)} = K(\xi) \|u\|_B^{2\theta} \|u\|_A^{2(1-\theta)}.
 \end{aligned}$$

Let us check that $K(\xi)$ is sharp. We first assume that the supremum in (2.36) is the maximum which is achieved at $\lambda = \lambda_*$. Then,

$$0 = \frac{d}{d\lambda} (\lambda^\theta f_\xi(\lambda))|_{\lambda=\lambda_*} = \lambda_*^\theta (f'_\xi(\lambda_*) + \theta \lambda_*^{-1} f_\xi(\lambda_*))$$

and $\lambda f'_\xi(\lambda_*) + \theta f_\xi(\lambda_*) = 0$. Therefore, due to (2.32),

$$D(\lambda_*) = \frac{h_\xi(\lambda_*)}{g_\xi(\lambda_*)} = -\frac{f_\xi(\lambda_*) + \lambda_* f'_\xi(\lambda_*)}{f'_\xi(\lambda_*)} = \frac{1-\theta}{\theta} \lambda_*.$$

Thus, $\lambda_* = \frac{\theta}{1-\theta} \frac{\|G_{\lambda_*}(\xi, \cdot)\|_A^2}{\|G_{\lambda_*}(\xi, \cdot)\|_B^2}$ and all inequalities in (2.37) become equalities if we take $u(x) = G_{\lambda_*}(\xi, x)$, so we have the exact extremal function in that case.

Since $\lambda^\theta G_\lambda(\xi, \xi) \rightarrow 0$ as $\lambda \rightarrow 0$, we only need to consider the case when the supremum in (2.36) is achieved as $\lambda \rightarrow \infty$. Then, two alternative cases are possible:

1) there are sequence $\{\lambda_k\}_{k=1}^\infty$ of *local* maximums such that

$$\lambda_k^\theta G_{\lambda_k}(\xi, \xi) \rightarrow \sup_{\lambda > 0} \{ \lambda^\theta G_\lambda(\xi, \xi) \}$$

Since the derivative vanishes at local maximums, then arguing as before, we see that the sequence of conditional extremals $u_n(x) := G_{\lambda_n}(\xi, x)$ does not allow us to take the constant K strictly less than $K(\xi)$ and (2.36) is sharp.

2) The function $\lambda^\theta G_\lambda(\xi, \xi)$ is eventually monotone increasing as $\lambda \rightarrow \infty$. Then the limit

$$(2.38) \quad G_\infty := \lim_{\lambda \rightarrow \infty} \lambda^\theta G_\lambda(\xi, \xi)$$

exists and is strictly positive. Using the fact that the derivative is integrable we can find sequences $\lambda_k \rightarrow \infty$ and $\varepsilon_k \rightarrow 0$ such that

$$\frac{d}{d\lambda} (\lambda^\theta f_\xi(\lambda))|_{\lambda=\lambda_k} = \varepsilon_k \lambda_k^{-1}.$$

This, together with (2.38) gives

$$\lambda_k^\theta f_\xi(\lambda_k) = G_\infty + o_{\lambda \rightarrow \infty}(1), \quad \lambda_k^{1+\theta} f'_\xi(\lambda_k) = -\theta G_\infty + o_{\lambda \rightarrow \infty}(1).$$

Therefore,

$$(2.39) \quad \frac{D(\lambda_k)}{\lambda_k} = \frac{1-\theta}{\theta} + o_{\lambda \rightarrow \infty}(1).$$

Finally, taking $u_k(x) := G_{\lambda_k}(\xi, x)$, after straightforward transformations we see that

$$(2.40) \quad \frac{|u_k(\xi)|^2}{\|u_k\|_B^{2\theta} \|u_k\|_A^{2(1-\theta)}} = \left(\lambda_k^\theta G_{\lambda_k} \right) \cdot \left(\frac{\lambda_k}{D(\lambda_k)} \right)^{1-\theta} \cdot \left(1 + \frac{D(\lambda_k)}{\lambda_k} \right)$$

Passing to the limit $k \rightarrow \infty$ and using (2.39), we obtain exactly $K(\xi)$ in the right-hand side and verify that $K(\xi)$ is sharp in the second case as well.

To complete the proof it remains to show that if there exists an extremal function u_* in (2.35), (2.36), then the supremum with respect to λ in (2.36) is attained at a finite point.

Using the elementary identity for positive a, b

$$\lambda_*^\theta a^{2(1-\theta)} b^{2\theta} = \theta^\theta (1-\theta)^{(1-\theta)} (a^2 + \lambda_* b^2), \quad \lambda_* = \frac{\theta}{1-\theta} \frac{a^2}{b^2},$$

we have for the extremal function u_* the equality

$$\begin{aligned} u_*(\xi) &= K(\xi) \|u\|_B^{2\theta} \|u\|_A^{2(1-\theta)} = K(\xi) \frac{\theta^\theta (1-\theta)^{(1-\theta)}}{\lambda_*^\theta} (\|u_*\|_A^2 + \lambda_* \|u_*\|_B^2) = \\ &= \sup_{\lambda > 0} \left\{ \lambda^\theta G_\lambda(\xi, \xi) \right\} \frac{1}{\lambda_*^\theta} (\|u_*\|_A^2 + \lambda_* \|u_*\|_B^2), \quad \text{where } \lambda_* = \frac{\theta}{1-\theta} \frac{\|u_*\|_A^2}{\|u_*\|_B^2}. \end{aligned}$$

Lemma 2.1 now gives that necessarily $u_*(x) = G_{\lambda_*}(x, \xi)$ and

$$\sup_{\lambda > 0} \left\{ \lambda^\theta G_\lambda(\xi, \xi) \right\} \frac{1}{\lambda_*^\theta} = G_{\lambda_*}(\xi, \xi), \quad \text{or} \quad \sup_{\lambda > 0} \left\{ \lambda^\theta G_\lambda(\xi, \xi) \right\} = \lambda_*^\theta G_{\lambda_*}(\xi, \xi).$$

The proof is complete. \square

2.5. Asymptotic expansions for big λ . In this subsection, we derive some useful formulas for the function $\mathbb{V}(\xi, D)$ when $D \rightarrow \infty$. To this end, we need to know the asymptotic behavior of the Green's function $f_\xi(\lambda) = G_\lambda(\xi, \xi)$. It is not difficult to see using the localization and frozen coefficients technique that the limit (2.38) exists and is strictly positive, so the leading term in the asymptotic expansions of G_λ is known. However, the further terms in the asymptotic expansion seems problem dependent and we do not know the general formulas for them. By this reason, we just assume that

$$(2.41) \quad f_\xi(\lambda) = \lambda^{-1} \left(g_1 \lambda^{1-\theta} + g_2 + g_3 \lambda^{\theta-1} + o(\lambda^{\theta-1}) \right),$$

where $g_1 = g_1(\xi) > 0$ and $g_2 = g_2(\xi)$, $g_3 = g_3(\xi)$ are some given numbers, and that we are able to differentiate the expansions (2.41) with respect to λ . This assumption will be satisfied in most part of our applications. Then, the following result holds.

Proposition 2.6. *Let the above assumptions hold and let, in addition, the asymptotic expansions (2.41) be true. Then the following asymptotic expansion holds as $D \rightarrow \infty$:*

$$(2.42) \quad \mathbb{V}(\xi, D) = g_1 S D^{1-\theta} + g_2 \frac{1}{\theta} - \frac{1}{2} S^{-1} \frac{g_2^2(1-\theta) - 2\theta g_1 g_3}{\theta^3 g_1} D^{\theta-1} + o(D^{\theta-1}), \quad S := \frac{1}{\theta^\theta (1-\theta)^{1-\theta}}.$$

Proof. The proof of this proposition is a straightforward (although rather technical) computation. Setting for brevity $g_1 = a$, $g_2 = b$, $g_3 = c$ we have

$$\begin{aligned} f(\lambda) &= a\lambda^{-\theta} + b\lambda^{-1} + c\lambda^{-(2-\theta)} + o(\lambda^{-(2-\theta)}), \\ g(\lambda) &= a\theta\lambda^{-(1+\theta)} + b\lambda^{-2} + c(2-\theta)\lambda^{-(3-\theta)} + o(\lambda^{-(3-\theta)}), \\ h(\lambda) &= f(\lambda) - \lambda g(\lambda) = a(1-\theta)\lambda^{-\theta} - c(1-\theta)\lambda^{-(2-\theta)} + o(\lambda^{-(2-\theta)}). \end{aligned}$$

Next, we find the asymptotics, as $D \rightarrow \infty$, of the (unique) solution $\lambda = \lambda(D)$ of the first equation in (2.31):

$$\begin{aligned} D(\lambda) &= \frac{h(\lambda)}{g(\lambda)} = \frac{\lambda(a(1-\theta) - c(1-\theta)\lambda^{-(2-2\theta)} + \dots)}{a\theta + b\lambda^{-(1-\theta)} + c(2-\theta)\lambda^{-(2-2\theta)} + \dots} = \\ &= \frac{1-\theta}{\theta} \left(\lambda - \frac{b}{a\theta} \lambda^\theta + \frac{b^2 - 2\theta ac}{a^2 \theta^2} \lambda^{-1+2\theta} + \dots \right), \end{aligned}$$

or

$$(2.43) \quad \lambda - A\lambda^\theta + B\lambda^{-1+2\theta} + \dots = \delta := D \frac{\theta}{1-\theta}, \quad A = \frac{b}{a\theta}, \quad B = \frac{b^2 - 2\theta ac}{a^2\theta^2}.$$

The unique large solution of this equation has the asymptotics as $\delta \rightarrow \infty$

$$\lambda(\delta) = \delta + A\delta^\theta + C\delta^{-1+2\theta} + \dots,$$

where we find C by substituting the last expression into (2.43), which gives

$$C = \theta A^2 - B = \frac{-b^2(1-\theta) + 2ac\theta}{a^2\theta^2},$$

or, finally,

$$(2.44) \quad \lambda(D) = rD + sD^\theta + tD^{2\theta-1} + \dots,$$

where

$$r = \frac{\theta}{1-\theta}, \quad s = \frac{b}{a} \frac{1}{\theta^{1-\theta}(1-\theta)^\theta}, \quad t = \frac{2ac\theta - (1-\theta)b^2}{a^2\theta^{3-2\theta}(1-\theta)^{2\theta-1}}.$$

It remains to substitute (2.44) into $\mathbb{V} = f^2/g = D \cdot f^2/h$, for which we have the expansion

$$(2.45) \quad \frac{f(\lambda)^2}{h(\lambda)} = \frac{a}{1-\theta} \lambda^{-\theta} + \frac{2b}{1-\theta} \lambda^{-1} + \frac{b^2 + 3ac}{(1-\theta)a} \lambda^{\theta-2} + \dots$$

For each power we obtain from (2.44), respectively,

$$\begin{aligned} \lambda(D)^{-\theta} &= \frac{1}{r^\theta} D^{-\theta} - \frac{\theta s}{r^{1+\theta}} D^{-1} + \frac{\theta(\theta+1)s^2 - 2\theta tr}{2r^{2+\theta}} D^{\theta-2} + \dots, \\ \lambda(D)^{-1} &= \frac{1}{r} D^{-1} - \frac{s}{r^2} D^{\theta-2} + \dots, \\ \lambda(D)^{\theta-2} &= \frac{1}{r^{2-\theta}} D^{\theta-2} + \dots \end{aligned}$$

Substituting this into (2.45), multiplying by D , we obtain after quite a few miraculous cancellations the asymptotic expansion (2.42). \square

Remark 2.7. We see that, in particular, if for some $\xi \in \mathcal{M}$ the third term in (2.42) is negative (this is always the case when $g_3 \leq 0$), then

$$(2.46) \quad \mathbb{V}(\xi, D) < g_1 S D^{1-\theta} + \frac{g_2}{\theta},$$

for large D . Then, using the numerics which is reliable for relatively small D , we will show that, in some cases, inequality (2.46) holds for *all* values of D . This will give us the improved version of (2.9):

$$(2.47) \quad |u(\xi)|^2 \leq g_1 S \|u\|_B^\theta \|u\|_A^{1-\theta} + \frac{g_2}{\theta} \|u\|_B^2$$

with best possible constants.

We also mention an interesting fact that

$$D'(\lambda) = \left(\frac{f_\xi(\lambda) + \lambda f'_\xi(\lambda)}{-f'_\xi(\lambda)} \right)' = \frac{2(f'_\xi(\lambda))^2 - f''_\xi(\lambda)f(\lambda)}{(f'_\lambda(\xi))^2}$$

and, therefore, the positivity of $D'(\lambda)$ proved in Lemma 2.2 is equivalent to the strict convexity of the function $\lambda \rightarrow \frac{1}{f_\xi(\lambda)}$.

2.6. Variational characterisation of $\mathbb{V}(\xi, \lambda)$. In this subsection, we give a simple, but very useful description of $\mathbb{V}(\xi, \lambda)$ in terms of the Green's function $G_\lambda(\xi, \xi)$ which does not involve the derivatives in λ and which allows us in many cases to prove inequality (2.46) *analytically* for all admissible values of D . Namely, the following theorem holds.

Theorem 2.8. *Let the assumptions of Theorem 2.3 hold. Then, the function $\mathbb{V}(\xi, D)$ defined in (2.29) can be expressed as follows:*

$$(2.48) \quad \mathbb{V}(\xi, D) = \inf_{\lambda \in [-\lambda_0, \infty)} \{(\lambda + D)G_\lambda(\xi, \xi)\},$$

where $D \in [\lambda_0, \infty)$ and the Green's function $G_\lambda(\xi, \xi)$ is defined in (2.14).

Proof. We first note that, due to the results obtained above the function $\lambda \rightarrow (\lambda + D)G_\lambda(\xi, \xi)$ tends to $+\infty$ when $\lambda \rightarrow \infty$ or $\lambda \rightarrow -\lambda_0$ for every $D \in (\lambda_0, \infty)$. Thus, the infimum in (2.48) is achieved at some point $\lambda_0 = \lambda_0(D)$ inside the interval. Obviously, this λ_0 solves the equation

$$G_\lambda(\xi, \xi) + (D + \lambda) \frac{d}{d\lambda} G_\lambda(\xi, \xi) = 0, \text{ or } D = -\frac{\lambda \frac{d}{d\lambda} G_\lambda(\xi, \xi) + G_\lambda(\xi, \xi)}{\frac{d}{d\lambda} G_\lambda(\xi, \xi)} = \frac{h_\xi(\lambda)}{g_\xi(\lambda)},$$

see (2.30), (2.31) and (2.32). Thus, the equation on the minimal value $\lambda_0(D)$ coincides with equation (2.19) for $\lambda(D)$ which has a unique solution and, therefore, the minimum in (2.48) is achieved exactly at $\lambda_0 = \lambda(D)$. It remains to check that the value at this point is exactly $\mathbb{V}(\xi, \lambda)$. To this end, we note that at the extremal point the numbers D and λ satisfy the first equation of (2.31) and, at that point

$$(\lambda + D)G_\lambda(\xi, \xi) = \frac{f_\lambda(\xi)(h_\xi(\lambda) + \lambda g_\xi(\lambda))}{g_\xi(\lambda)} = \frac{f_\xi(\lambda)^2}{g_\xi(\lambda)} = \mathbb{V}(\xi, D),$$

where we have used (2.31). Theorem 2.8 is proved. \square

2.7. Generalizations. Here we briefly discuss several possibilities to relax the assumptions on the manifold \mathcal{M} and operators A and B .

I) The operators A and B should not necessarily be elliptic *differential* operators. All the theory works word for word if we assume that A and B are elliptic *pseudo-differential* operators (for instance, $A = (-\Delta)^m$ and $B = (-\Delta)^l$ where Δ is the Laplace-Beltrami operator on \mathcal{M}) and, in particular, the numbers m and l may be not integers. This allow us to study the interpolation inequalities in fractional Sobolev spaces.

II) The theory can be naturally extended to the case where the manifold \mathcal{M} has a boundary. In this case, we need to assume that the elliptic operators A and B are endowed by the proper *boundary conditions* and these boundary conditions are chosen in such way that

$$(2.49) \quad \mathcal{D}(A) \subset \mathcal{D}(B),$$

where $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are the domains of the operators A and B . It is not difficult to see that under this extra assumption(s) the above developed theory remains true for manifolds with boundary as well.

III) The spaces \bar{H}^m may be further restricted, for instance, we may consider not all functions, say, in a disk, but only radially symmetric ones. If the operators A and B are also radially symmetric, all the theory works in this case as well.

IV) The above theory works in many cases where the manifold \mathcal{M} is *not compact*. The only problem here is that, unlike the compact case, the Green's functions $G_\lambda(x, \xi)$ may have bad behavior as $x, \xi \rightarrow \infty$ and as a result, the integrals used above may not have sense. So, in general theory, one should be accurate with the extra assumptions on the non-compact part of \mathcal{M} and operators A and B as well as with the possible continuous spectrum of A and B . However, in all our “non-compact” applications, these questions will be obvious and transparent,

so in order to avoid the technicalities, we do not present here any “general theory” for the non-compact case.

3. PART II. EXAMPLES AND APPLICATIONS

3.1. The case of \mathbb{R}^n . To illustrate our method we consider the simplest case when $M = \mathbb{R}^n$, and let l and m satisfy $-\infty < l < n/2 < m < \infty$, so that $0 < \theta = \frac{2m-n}{2(m-l)} < 1$.

Theorem 3.1. *The following inequality holds*

$$(3.1) \quad \|u\|_\infty^2 \leq c_{\mathbb{R}^n}(l, m) \|(-\Delta)^{l/2} u\|^{2\theta} \|(-\Delta)^{m/2} u\|^{2(1-\theta)},$$

where the sharp constant $c_{\mathbb{R}^n}(l, m)$ is

$$(3.2) \quad c_{\mathbb{R}^n}(l, m) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{d\xi}{\theta|\xi|^{2l} + (1-\theta)|\xi|^{2m}} = \frac{\sigma(n)\pi}{(2\pi)^n 2(m-l)\theta^\theta(1-\theta)^{1-\theta} \sin \pi\theta}.$$

Proof. Using the Fourier transform $\widehat{f}(\xi) = \mathcal{F}(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi x} f(x) dx$, for $\lambda > 0$

$$(3.3) \quad \begin{aligned} \|u\|_\infty^2 &\leq (2\pi)^{-n} \left(\int_{\mathbb{R}^n} |\widehat{u}(\xi)| d\xi \right)^2 \leq \\ &(2\pi)^{-n} \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 \left(\lambda\theta|\xi|^{2l} + (1-\theta)|\xi|^{2m} \right) d\xi \int_{\mathbb{R}^n} \frac{d\xi}{\lambda\theta|\xi|^{2l} + (1-\theta)|\xi|^{2m}} = \\ &(2\pi)^{-n} \int_{\mathbb{R}^n} \frac{d\xi}{\lambda\theta|\xi|^{2l} + (1-\theta)|\xi|^{2m}} \cdot \left(\lambda\theta \|(-\Delta)^{l/2} u\|^2 + (1-\theta) \|(-\Delta)^{m/2} u\|^2 \right). \end{aligned}$$

Next, setting $\lambda = \lambda_* = \|(-\Delta)^{m/2} u\|^2 / \|(-\Delta)^{l/2} u\|^2$ we have

$$(3.4) \quad \lambda_* \theta \|(-\Delta)^{l/2} u\|^2 + (1-\theta) \|(-\Delta)^{m/2} u\|^2 = \lambda_*^\theta \|(-\Delta)^{l/2} u\|^{2\theta} \|(-\Delta)^{m/2} u\|^{2(1-\theta)},$$

which gives

$$\|u\|_\infty^2 \leq (2\pi)^{-n} \left[\lambda_*^\theta \int_{\mathbb{R}^n} \frac{d\xi}{\lambda_* \theta |\xi|^{2l} + (1-\theta) |\xi|^{2m}} \right] \|(-\Delta)^{l/2} u\|^{2\theta} \|(-\Delta)^{m/2} u\|^{2(1-\theta)}.$$

The expression in brackets is, in fact, independent of λ_* , which gives (3.2). The fact that the constant is sharp can be verified by substituting

$$(3.5) \quad u_\lambda(x) = \mathcal{F}^{-1} \left((\lambda\theta_1|\xi|^{2l} + \theta_2|\xi|^{2m})^{-1} \right),$$

and calculating the corresponding integrals. A simpler way, however, is to observe first that for $u_\lambda(x)$ all inequalities in (3.3) become equalities. Next, differentiating the expression in brackets with respect to λ , one can see that $\lambda = \|(-\Delta)^{l/2} u_\lambda\|^2 / \|(-\Delta)^{m/2} u_\lambda\|^2$, which proves that (3.1) for $u = u_\lambda$ becomes an equality. \square

Remark 3.2. Of course, we get the same result by applying Theorem 2.3. Here

$$\mathbb{A}(\lambda) = (-\Delta)^m + \lambda(-\Delta)^l, \quad G_\lambda(x, \xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{i\eta(x-\xi)} d\eta}{|\eta|^{2m} + \lambda|\eta|^{2l}}$$

and

$$G_\lambda(\xi, \xi) = \lambda^{-\theta} \frac{\sigma(n)\pi}{(2\pi)^n 2(m-l) \sin \pi\theta}.$$

Corollary 3.3. *For any $D > 0$ the maximization problem*

$$(3.6) \quad \mathbb{V}(D) := \sup \left\{ |u(0)|^2 : \|(-\Delta)^{l/2} u\|^2 = 1, \|(-\Delta)^{m/2} u\|^2 = D \right\}$$

has the solution

$$(3.7) \quad \mathbb{V}(D) = c_{\mathbb{R}^n}(l, m) D^{1-\theta}.$$

The unique extremal function is

$$U_D = \frac{u_D}{\|(-\Delta)^{l/2} u_D\|},$$

see (3.5).

Remark 3.4. The integral was calculated using the formula

$$(3.8) \quad \int_0^\infty \frac{x^m}{(1+x^k)^l} dx = \frac{1}{k} B\left(\frac{m+1}{k}, l - \frac{m+1}{k}\right),$$

which will also be helpful in what follows.

Remark 3.5. In the 1D case the constant $c_{\mathbb{R}}(l, m)$ was found in [23].

3.2. Symmetric manifolds. In this section, we discuss the case where the underlying manifold \mathcal{M} is *symmetric* and the operators A and B are invariant with respect to the symmetry group. As a consequence of this, the Green's function $G_\lambda(x, \xi)$ introduced in (2.14) depends only on $x - \xi$ (since every two points on \mathcal{M} can be identified by the proper symmetry map). By this reason, the key function $G_\lambda(\xi, \xi)$ is in fact independent of ξ and is a function of one variable λ :

$$(3.9) \quad G_\lambda(\xi, \xi) = G(\lambda).$$

This observation simplifies greatly the analysis and allows us in many cases to compute explicitly the best constants in the appropriate interpolation inequalities. We restrict ourselves below only to consider the two model examples: tori and spheres although the developed technique is applicable to other symmetric manifolds as well.

3.2.1. The tori. We start with the case of n -dimensional torus $\mathbb{T}^n := [-\pi, \pi]^n$ and the inequalities of the form

$$(3.10) \quad \|u\|_{L_\infty(\mathbb{T}^n)} \leq C_{l,n,m} \|(-\Delta)^{l/2} u\|_{L_2(\mathbb{T}^n)}^\theta \|(-\Delta)^{m/2} u\|_{L_2(\mathbb{T}^n)}^{1-\theta}$$

for the periodic functions $u \in H^m(\mathbb{T}^n)$ with zero mean ($\langle u \rangle := \int_{\mathbb{T}^n} u(x) dx = 0$). In that case, $A := (-\Delta)^m$, $B = (-\Delta)^l$, $\ker A = \ker B = \{\text{const}\}$ and all of the assumptions of the above developed abstract theory are satisfied if $l < \frac{n}{2} < m$. In particular,

$$(3.11) \quad \bar{H}^m = H^m(\mathbb{T}^n) \cap \{\langle u \rangle = 0\}.$$

The Green's function of the operator $\mathbb{A}(\lambda) = (-\Delta)^m + \lambda(-\Delta)^l$ on the torus \mathbb{T}^n (with zero mean) can be found by expanding it into the multi-dimensional Fourier series. This gives

$$(3.12) \quad G_\lambda(x, \xi) = \frac{1}{(2\pi)^n} \sum_{k \in \mathbb{Z}_0^n} \frac{e^{ik(x-\xi)}}{|k|^{2m} + \lambda|k|^{2l}}, \quad G_\lambda(\xi, \xi) = G(\lambda) = \frac{1}{(2\pi)^n} \sum_{k \in \mathbb{Z}_0^n} \frac{1}{|k|^{2m} + \lambda|k|^{2l}},$$

where $k = (k_1, \dots, k_n)$ is the multi-index, $|k|^2 = k_1^2 + \dots + k_n^2$ and the summation holds for all multi-indexes $k = (k_1, \dots, k_n) \neq (0, \dots, 0)$: $\mathbb{Z}_0^n = \mathbb{Z}^n \setminus \{0\}$.

3.2.1.1. The case $l = 0$. The particular case $l = 0$ has been studied in [2]. In that case, the asymptotic behavior of $G(\lambda)$ as $\lambda \rightarrow \infty$ can be analyzed in a straightforward way using the Poisson summation formula (see, e.g., [22]):

$$(3.13) \quad \sum_{m \in \mathbb{Z}^n} f(m/\mu) = (2\pi)^{n/2} \mu^n \sum_{m \in \mathbb{Z}^n} \hat{f}(2\pi m \mu),$$

where $\mathcal{F}(f)(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} dx$ is the Fourier transform and $\mu > 0$.

Namely, let

$$\Phi_\lambda(x) = (2\pi)^{-n/2} \mathcal{F}^{-1}(1/(|\xi|^{2m} + \lambda)) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{i\xi \cdot x} d\xi}{|\xi|^{2m} + \lambda}$$

be the fundamental solution of the differential operator $\mathbb{A}(\lambda)$ in the whole space \mathbb{R}^n . Then (3.13) gives

$$(3.14) \quad G(\lambda) + \frac{1}{(2\pi)^n} \lambda^{-1} = \frac{1}{(2\pi)^n} \sum_{k \in \mathbb{Z}^n} \frac{1}{|k|^{2m} + \lambda} = \sum_{k \in \mathbb{Z}^n} \Phi_\lambda(2\pi k).$$

Furthermore, due to scaling invariance, $\Phi_\lambda(x) = \lambda^{-1} \lambda^{n/(2m)} \Phi_1(\lambda^{1/(2m)} x)$ and, due to the analyticity of the function $\xi \rightarrow \frac{1}{|\xi|^{2m+1}}$, we have $|\Phi_1(x)| \leq C e^{-C_m |x|}$ for some positive C and C_m . Therefore, (3.14) reads

$$(3.15) \quad G(\lambda) = \lambda^{-1} \lambda^{n/(2m)} \Phi_1(0) - \frac{1}{(2\pi)^n} \lambda^{-1} + O(e^{-C_m \lambda^{1/(2m)}}).$$

Finally, computing the table integral $\Phi_1(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{d\xi}{|\xi|^{2m+1}}$, we end up with

$$(3.16) \quad G(\lambda) = (2\pi)^{-n} \lambda^{-1} \left(\frac{\pi \sigma(n)}{2m \sin(\frac{\pi n}{2m})} \lambda^{\frac{n}{2m}} - 1 \right) + O(e^{-C_m \lambda^{1/(2m)}}),$$

where $\sigma(n) = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the $(n-1)$ -dimensional unit sphere. Note that $1 - \theta = n/2m$ now, so (3.16) has the form of (2.41) with $g_3 = 0$ and, due to 2.6, we have

$$(3.17) \quad \mathbb{V}(D) = c_n(m) D^{n/(2m)} - k_n(m) - l_n(m) D^{-n/(2m)} + O(D^{-n/m}),$$

where

$$(3.18) \quad c_n(m) := c_{\mathbb{R}^n}(0, m) = \frac{\pi \sigma(n)}{(2\pi)^n n^{n/(2m)} (2m - n)^{1-n/(2m)} \sin \frac{\pi n}{2m}}, \quad k_n(m) := \frac{2m}{(2\pi)^n (2m - n)},$$

$$l_n(m) := \frac{2n^{1+n/(2m)} m^2 \sin \frac{\pi n}{2m}}{(2\pi)^n \pi \sigma(n) (2n - m)^{2+n/(2m)}},$$

see [2] for the details. Thus, since $l_n(m) > 0$,

$$(3.19) \quad \mathbb{V}(D) \leq c_n(m) D^{n/(2m)} - k_n(m)$$

for large D and using the fact that $D \rightarrow \mathbb{V}(D)$ is continuous, we conclude that

$$(3.20) \quad K_n(m) := \sup_{D \in [1, \infty)} \{c_n(m) D^{n/(2m)} - \mathbb{V}(D)\} < \infty$$

and

$$(3.21) \quad \mathbb{V}(D) \leq c_n(m) D^{n/(2m)} - K_n(m)$$

already for all $D \geq 1$. Thus, according to (2.11), we have established the following result of [2].

Theorem 3.6. *The following inequality holds for all $u \in \bar{H}^m(\mathbb{T}^n)$:*

$$(3.22) \quad \|u\|_{L^\infty(\mathbb{T}^n)}^2 \leq c_n(m) \|u\|_{L_2(\mathbb{T}^n)}^{2-n/m} \|(-\Delta)^{m/2} u\|_{L_2(\mathbb{T}^n)}^{n/m} - K_n(m) \|u\|_{L_2(\mathbb{T}^n)}^2,$$

where $c_n(m)$ and $K_n(m) \leq k_n(m)$ are sharp and are determined by (3.18) and (3.20) respectively.

We see that the constant $c_n(m)$ here coincides with the best constant in the analogous inequality on the whole \mathbb{R}^n , see (3.2). In contrast to that, the sharp constant $K_n(m)$ is unlikely to be expressed analytically for all n and m since, according to the definition (3.20), we need to know the function $V(D)$ not only for large D , but for all $D \geq 1$ in order to compute it. As shown in [2], the asymptotic expansion (3.17) works for *very large* D only (if m is large enough) and for the intermediate values of D this function is *oscillatory* (which can be explained by studying the limit $m \rightarrow \infty$, see [2] for the details).

Moreover, there is a strong difference between the case $n = 1$ and the multi-dimensional case $n > 1$. In the first case, as shown in [2], we always have $K_1(m) > 0$, so the lower order term in

(3.22) *improves* the classical interpolation inequality. However, in the multi-dimensional case, this constant become strictly negative for sufficiently large m (for instance, for $m > 9$ if $n = 2$ and for $m > 6$ for $n = 3$). Therefore, in that case, the lower order corrector becomes *positive* and necessary for the validity of the interpolation inequality. In other words, if we are interested only in the classical interpolation inequality on the torus (without the lower order correctors), we have to *increase* the constant in comparison with the case of \mathbb{R}^n .

Nevertheless, as shown in [2] with the help of *numerics* there are 3 particular cases where $K_n(m) = k_n(m)$ and as a consequence, all constants in (3.22) can be found analytically. That are

1) The case $n = 1$ and $m = 1$: the inequality

$$(3.23) \quad \|u\|_{L_\infty(\mathbb{T}^1)}^2 \leq \|u\|_{L_2(\mathbb{T}^1)} \|u'\|_{L_2(\mathbb{T}^1)} - \frac{1}{\pi} \|u\|_{L_2(\mathbb{T}^1)}^2$$

holds for all 2π -periodic functions with zero mean;

2) The case $n = 1$ and $m = 2$: the inequality

$$(3.24) \quad \|u\|_{L_\infty(\mathbb{T}^1)}^2 \leq \frac{\sqrt{2}}{\sqrt[4]{27}} \|u\|_{L_2(\mathbb{T}^1)}^{3/2} \|u''\|_{L_2(\mathbb{T}^1)}^{1/2} - \frac{2}{3\pi} \|u\|_{L_2(\mathbb{T}^1)}^2$$

holds for all 2π -periodic functions with zero mean;

3) The case $n = 2$ and $m = 2$: the inequality

$$(3.25) \quad \|u\|_{L_\infty(\mathbb{T}^2)}^2 \leq \frac{1}{4} \|u\|_{L_2(\mathbb{T}^2)} \|\Delta u\|_{L_2(\mathbb{T}^2)} - \frac{1}{2\pi^2} \|u\|_{L_2(\mathbb{T}^2)}^2$$

holds for all $2\pi \times 2\pi$ -periodic functions with zero mean. As the numerics suggests, that are the only cases (at least with integer m) when the constant $K_n(m) = k_n(m)$, see [2].

Based on the technique developed above, we give below the purely analytic proof of the first two inequalities.

Theorem 3.7. *The interpolation inequalities (3.23) and (3.24) hold for all 2π -periodic functions with zero mean and all constants in those inequalities are sharp.*

Proof. We first consider (3.23). Since $\Phi_\lambda(x) = (2\pi)^{-1/2} \mathcal{F}^{-1}((\xi^2 + \lambda)^{-1}) = \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}|x|}$ for all $\lambda > 0$, according to (3.14) and summing the geometric progression, we have

$$(3.26) \quad G(\lambda) = \sum_{k \in \mathbb{Z}} \frac{1}{2\sqrt{\lambda}} e^{-2\pi\sqrt{\lambda}|k|} - \frac{1}{2\pi\lambda} = \frac{1}{2\pi} \frac{\pi\sqrt{\lambda} \coth(\pi\sqrt{\lambda}) - 1}{\lambda}.$$

Thus, in view of Theorem 2.8, for $D \geq 1$,

$$(3.27) \quad \mathbb{V}(D) = \min_{\lambda \geq -1} \{(\lambda + D)G(\lambda)\} \leq \{(\lambda + D)G(\lambda)\}_{|\lambda=D-1/2} = (2D - 1/2)G(D - 1/2),$$

where we have replaced minimum with respect to λ by the value at $\lambda = D - 1/2$. Thus, we only need to prove that

$$(3.28) \quad (2D - 1/2)G(D - 1/2) - D^{1/2} + \frac{1}{\pi} = \frac{1}{4\pi(2D - 1)} \left(4\pi D \sqrt{4D - 2} \cdot \coth \alpha - \pi \sqrt{4D - 2} \cdot \coth \alpha - 2 - 8\pi D^{3/2} + 4\pi \sqrt{D} \right) < 0$$

for all $D \geq 1$, where we set for brevity $\alpha := \frac{\pi\sqrt{4D-2}}{2}$. To simplify the expression on the right-hand side of (3.19), we use that

$$\sqrt{4D - 2} < 2\sqrt{D} - \frac{1}{2}D^{-1/2} - \frac{1}{16}D^{-3/2}, \quad D \geq 1$$

(this inequality is obtained by expanding $\sqrt{1-(2D)^{-1}}$ in Taylor series and noting that all dropped out terms there are negative). Using also that $\coth \alpha \geq 1$ in the negative terms, we end up with

$$\begin{aligned}
 (3.29) \quad \mathbb{V}(D) - D^{1/2} + \frac{1}{\pi} &\leq \frac{1}{4\pi(2D-1)} \left(4\pi D(8D^{1/2} - 2D^{-1/2} - D^{-3/2}/4) \cdot \coth \alpha - \right. \\
 &\quad \left. \pi\sqrt{4D-2} \cdot \coth \alpha - 2 - 8\pi D^{3/2} + 4\pi\sqrt{D} \right) = \\
 &= \frac{1}{16\pi D^{1/2}(2D-1)} \left(32\pi D^2(\coth \alpha - 1) - \right. \\
 &\quad \left. 8\pi D \coth \alpha - \pi \coth \alpha - 4\pi\sqrt{4D-2}\sqrt{D} - 8\sqrt{D} + 16\pi D \right) < \\
 &< \frac{1}{16\pi D^{1/2}(2D-1)} \left(\frac{64\pi D^2}{e^{2\alpha} - 1} - 8\pi D - \pi - 4\pi\sqrt{4D-2}\sqrt{D} - 8\sqrt{D} + 16\pi D \right) = \\
 &= \frac{1}{16\pi D^{1/2}(2D-1)} \left(\frac{64\pi D^2}{\exp(\pi\sqrt{4D-2}) - 1} + \frac{4\pi}{1 + \sqrt{1-(2D)^{-1}}} - \pi - 8\sqrt{D} \right).
 \end{aligned}$$

We note that the function $f(D) := \frac{64\pi D^2}{\exp(\pi\sqrt{4D-2}) - 1}$ is strictly decreasing when $D \geq 1$ since

$$f'(D) = -\frac{128\pi D (\sqrt{4D-2} - \sqrt{4D-2} \exp(\pi\sqrt{4D-2}) + \pi D \exp(\pi\sqrt{4D-2}))}{(-1 + \exp(\pi\sqrt{4D-2}))^2 \sqrt{4D-2}} < 0$$

for all $D \geq 1$. Analogously, the second term in the right-hand side of (3.29) and the last one are also strictly decreasing, so replacing them by the their maximal values at $D = 1$, we finally have

$$\begin{aligned}
 \mathbb{V}(D) - D^{1/2} + \frac{1}{\pi} &< \frac{1}{16\pi D^{1/2}(2D-1)} \left(\frac{64\pi}{\exp(\pi\sqrt{2}) - 1} + \frac{8\pi}{2 + \sqrt{2}} - \pi - 8 \right) = \\
 &= \frac{1}{16\pi D^{1/2}(2D-1)} \cdot (-1.3873\dots) < 0.
 \end{aligned}$$

Thus, inequality (3.23) is proved. The fact that all constants there are sharp follows from the previously established asymptotics (3.17):

$$\mathbb{V}(D) = D^{1/2} - \frac{1}{\pi} + O(D^{-1/2}) \quad \text{as } D \rightarrow \infty.$$

Before we turn to (3.24) we note that the general formula (2.44) gives in our case $\lambda(D) = D - \frac{2}{\pi}\sqrt{D} + \dots$. Other choices of λ in the substitution in (3.27) will do, for example, $\lambda = D - 1$, $\lambda = D - \frac{2}{\pi}\sqrt{D}$. However, the choice $\lambda = D$ will not.

We now consider (3.24). Using contour integration (see [24, Section 3.3]) we can sum the series in $G(\lambda)$:

$$\sum_{n=1}^{\infty} \frac{1}{\mu^4 + n^4} = \frac{\pi\sqrt{2}}{4\mu^3} \frac{\sinh(\pi\sqrt{2}\mu) + \sin(\pi\sqrt{2}\mu)}{\cosh(\pi\sqrt{2}\mu) - \cos(\pi\sqrt{2}\mu)} - \frac{1}{2\mu^4},$$

and hence

$$(3.30) \quad G(\lambda) = \frac{2}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^4 + \lambda} = \frac{1}{\pi} \left(\frac{\pi\sqrt{2}}{4\lambda^{3/4}} \frac{\sinh(\pi\sqrt{2}\lambda^{1/4}) + \sin(\pi\sqrt{2}\lambda^{1/4})}{\cosh(\pi\sqrt{2}\lambda^{1/4}) - \cos(\pi\sqrt{2}\lambda^{1/4})} - \frac{1}{2\lambda} \right).$$

Since

$$\frac{\sinh \alpha + \sin \alpha}{\cosh \alpha - \cos \alpha} \leq \frac{\sinh \alpha + 1}{\cosh \alpha - 1} < 1 + 4.1e^{-\alpha} \quad \text{for } \alpha = \pi\sqrt{2}\lambda^{1/4} \geq \pi\sqrt{2} = 4.4428\dots,$$

it follows that

$$G(\lambda) < G_1(\lambda) := \frac{1}{\pi} \left(\frac{\pi\sqrt{2}}{4\lambda^{3/4}} \left(1 + 4.1e^{-\pi\sqrt{2}\lambda^{1/4}} \right) - \frac{1}{2\lambda} \right)$$

By Theorem 2.8, for $D \geq 1$, substituting $\lambda = 3D - 3/2$ we obtain

$$(3.31) \quad \mathbb{V}(D) = \min_{\lambda \geq -1} \{(\lambda + D)G(\lambda)\} \leq \\ \leq \{(\lambda + D)G(\lambda)\}_{\lambda=3(D-1/2)} = (4D - 3/2)G(3D - 3/2) < (4D - 3/2)G_1(3D - 3/2).$$

Therefore it suffices to show that

$$R(D) := (4D - 3/2)G_1(3D - 3/2) - \frac{\sqrt{2}}{\sqrt[4]{27}}D^{1/4} + \frac{2}{3\pi} \leq 0.$$

Setting $x := (3D - 3/2)^{1/4}$, $x \geq (3/2)^{1/4} = 1.1066 \dots$ we have

$$6\pi R(D) = (8x^4 + 3) \left(\frac{\pi\sqrt{2}}{4x^3} \left(1 + 4.1e^{-\pi\sqrt{2}x} \right) - \frac{1}{2x^4} \right) - 2\sqrt{2}\pi(x^4 + 3/2)^{1/4} + 4 = \\ = 2\pi\sqrt{2} \left(\left(x + \frac{3}{8x^3} - (x^4 + 3/2)^{1/4} \right) + \left(x + \frac{3}{8x^3} \right) 4.1e^{-\pi\sqrt{2}x} \right) - \frac{3}{2x^4} =: R(x).$$

The first term in parenthesis is $O(x^{-7})$, hence $R(x) < 0$ already for, say, $x \geq 2$. Instead of analyzing $R(x)$ near $x = (3/2)^{1/4}$ we show in Fig. 1 the graph of $R(x)$ in this region, so that $R(x) < 0$ for all x .

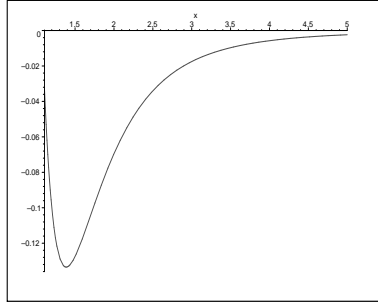


FIGURE 1. Graph of the function $R(x)$ on $x \in [(3/2)^{1/4}, 5]$.

In conclusion we point out that the general formula (2.44) gives $\lambda(D) = 3D - \frac{4\sqrt{2}}{\pi\sqrt[4]{3}}D^{3/4} + \dots$, which explains the choice of at least the leading term in the substitution $\lambda = 3D - 3/2$ that has been used above. The proof of the theorem is complete. \square

3.2.1.2. The case $l > 0$. The analysis of the Green's function (3.12) for general $l > 0$ is more delicate. Indeed, in this case, the function $k \rightarrow \frac{1}{|k|^{2m+\lambda}|k|^{2l}}$ has a singularity at $k = 0$. By this reason, the associated fundamental solution $\Phi_\lambda(x)$ is not rapidly decaying and the Poisson formula (3.14) becomes not essentially helpful. To overcome this difficulty, following [2], we introduce $\mu = \lambda^{-1}$ and rewrite (3.14) as follows

$$(3.32) \quad G(\lambda) = (2\pi)^{-n} \mu \sum_{k \in \mathbb{Z}_0^n} \frac{1}{k^{2l}(\mu|k|^{2(m-l)} + 1)} = \mu \tilde{G}(\mu).$$

Then, differentiating the function $\tilde{G}(\mu)$ s -times in μ , we get

$$(3.33) \quad \frac{d^s \tilde{G}(\mu)}{d\mu^s} = (-1)^s \Gamma(s+1) (2\pi)^{-n} \sum_{k \in \mathbb{Z}_0^n} \frac{|k|^{2(m-l)s-2l}}{(\mu|k|^{2(m-l)} + 1)^{s+1}}$$

and the function $k \rightarrow \frac{|k|^{2(m-l)s-l}}{(\mu|k|^{2(m-l)+1})^{s+1}}$ becomes regular at $k = 0$ if $s \geq \frac{l}{m-l}$ (and the rate of decay of this function as $k \rightarrow \infty$ remains $|k|^{2m}$ for all s). Thus, we may apply the Poisson summation formula in order to find the asymptotic behavior of (3.33) as $\mu \rightarrow 0+$ (analogously to (3.14)) which after the s -times integration will give us the asymptotics for the initial function $G(\lambda)$ up to s integration constants which should be determined using the alternative methods, see below.

We illustrate this method for the particular case $l = 1$ and $m = 2$ only. We also restrict ourselves to considering only the two ($n = 2$) and three ($n = 3$) dimensional cases although the general case can be analyzed in a similar way.

Let $n = 2$. Then, we need to differentiate $\tilde{G}(\mu)$ only once in order to remove the singularity at $k = 0$:

$$(3.34) \quad \frac{d\tilde{G}(\mu)}{d\mu} = -\frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}_0^2} \frac{1}{(1 + \mu|k|^2)^2}.$$

Applying the Poisson summation formula to this series, we get

$$(3.35) \quad \frac{d\tilde{G}(\mu)}{d\mu} = -\frac{1}{4\pi^2} \left(\frac{\pi}{\mu} - 1 + 2\pi^2 \mu^{-3/2} \sum_{k \in \mathbb{Z}_0^2} |k| K_1(2\pi \mu^{-1/2} |k|) \right),$$

where $K_n(z)$ is the modified Bessel function of order n , see [26] for the details. Integrating these series in μ and using the standard formulas for the integrals of Bessel functions, namely, $K_0'(x) = -K_1(x)$, we end up with the desired formula

$$(3.36) \quad \tilde{G}(\mu) = \frac{1}{4\pi^2} \left(\pi \log \frac{1}{\mu} + \beta + \mu - 8\pi \sum_{k \in \mathbb{Z}_0^2} K_0(2\pi \mu^{-1/2} |k|) \right),$$

where β is an integration constant. As shown in [2], based on the Hardy lattice formula

$$\sum_{k \in \mathbb{Z}_0^2} \frac{1}{|k|^{2s}} = 4\zeta(1+s)\beta(1+s),$$

where $\zeta(s)$ and $\beta(s)$ are the Riemann zeta and Dirichlet beta functions respectively,

$$(3.37) \quad \beta = \pi\gamma + 4\beta'(1) = \pi(2\gamma + 2\log 2 + 3\log \pi - 4\log \Gamma(1/4)),$$

where γ and $\Gamma(z)$ are the Euler-Mascheroni constant and Euler gamma function respectively. Thus, we have proved the following result.

Lemma 3.8. *Let $n = m = 2$ and $l = 1$. Then, at least for $\lambda > 0$, the Green's function (3.32) can be written as follows:*

$$(3.38) \quad G(\lambda) = \frac{1}{4\pi} \lambda^{-1} \log \lambda + \frac{\beta}{4\pi^2 \lambda} + \frac{1}{4\pi^2 \lambda^2} - \frac{1}{2\pi} \lambda^{-1} \sum_{k \in \mathbb{Z}_0^2} K_0(2\pi \lambda^{1/2} |k|) \leq \\ \leq \frac{1}{4\pi^2 \lambda} \left(\pi \log \lambda + \beta + \frac{1}{\lambda} \right),$$

where β is defined in (3.37). Moreover, the term containing the sum of Bessel functions in the middle part of (3.38) is exponentially small (of order $O(e^{-\pi \lambda^{1/2}})$) as $\lambda \rightarrow \infty$.

Note that the choice $n = m = 2$ and $l = 1$ corresponds to the limit case in (2.9) where the extra logarithmic term appears and which is usually called Brezis-Gallouet inequality (see [4]). Although the condition (2.8) is formally violated, most part of the theory developed above works

in that limit case as well. In particular, as shown in [2] based on (3.38), there exists a constant $L > 0$ such that

$$(3.39) \quad \mathbb{V}(D) \leq \frac{1}{4\pi} (\log D + \log(1 + \log D) + L), \quad D \geq 1,$$

where the constant $\frac{1}{4\pi}$ is sharp and the numerically found value of $L = 2.15627$. This leads to the improved Brezis-Galloet inequality in the form

$$(3.40) \quad \|u\|_{C(\mathbb{T}^2)}^2 \leq \frac{1}{4\pi} \|\nabla u\|_{L^2(\mathbb{T}^2)}^2 \left(\log \frac{\|\Delta u\|_{L^2(\mathbb{T}^2)}^2}{\|\nabla u\|_{L^2(\mathbb{T}^2)}^2} + \log \left(1 + \log \frac{\|\Delta u\|_{L^2(\mathbb{T}^2)}^2}{\|\nabla u\|_{L^2(\mathbb{T}^2)}^2} \right) + L \right).$$

Let us now consider the 3D case $n = 3$ with $m = 2$ and $l = 1$. Then,

$$(3.41) \quad \tilde{G}(\mu) = \frac{1}{8\pi^3} \sum_{k \in \mathbb{Z}_0^3} \frac{1}{|k|^2(1 + \mu|k|^2)}, \quad \frac{d\tilde{G}(\mu)}{d\mu} = -\frac{1}{8\pi^3} \sum_{k \in \mathbb{Z}_0^3} \frac{1}{(1 + \mu|k|^2)^2}$$

and, applying the Poisson summation formula (3.13) to the second sum, after the computation of the Fourier transform, namely, using (see [22])

$$(3.42) \quad \mathcal{F}(1/(1 + x^2)^2)(\xi) = \frac{\pi^2}{(2\pi)^{3/2}} e^{-|\xi|},$$

we end up with

$$(3.43) \quad \frac{d\tilde{G}(\mu)}{d\mu} = \frac{1}{8\pi^3} \left(1 - \frac{\pi^2}{\mu^{3/2}} \sum_{k \in \mathbb{Z}^3} e^{-2\pi\mu^{-1/2}|k|^{1/2}} \right),$$

and integrating this series in μ , we arrive at

$$(3.44) \quad \tilde{G}(\mu) = \frac{1}{8\pi^3} \left(2\pi^2\mu^{-1/2} + \beta_3 + \mu - \pi \sum_{k \in \mathbb{Z}_0^3} |k|^{-1/2} e^{-2\pi\mu^{-1/2}|k|^{1/2}} \right),$$

where β_3 is an integration constant which can be found numerically:

$$\beta_3 = -8.91363291758515127,$$

see the next section for more details on how to compute it. Thus, we have proved the following lemma.

Lemma 3.9. *Let $n = 3$, $m = 2$ and $l = 1$. Then, at least for $\lambda > 0$, the Green's function (3.32) can be written as follows:*

$$(3.45) \quad G(\lambda) = \frac{1}{8\pi^3\lambda} \left(2\pi^2\lambda^{1/2} + \beta_3 + \lambda^{-1} - \pi \sum_{k \in \mathbb{Z}_0^3} \frac{e^{-2\pi\lambda^{1/2}|k|^{1/2}}}{|k|^{1/2}} \right) \leq \frac{1}{8\pi^3\lambda} \left(2\pi^2\lambda^{1/2} + \beta_3 + \lambda^{-1} \right),$$

where β is defined in (3.37). Moreover, the term containing the sum in the middle part of (3.45) is exponentially small (of order $O(e^{-\pi\lambda^{1/2}})$ as $\lambda \rightarrow \infty$).

Thus, the Green's function $G(\lambda)$ satisfies the assumptions of Proposition 2.6 with $\theta = 1/2$ and $g_1, g_2, g_3 = (8\pi^3)^{-1}(2\pi^2, \beta_3, 1)$, respectively. Therefore,

$$(3.46) \quad \mathbb{V}(D) = \frac{1}{8\pi^3} \left(4\pi^2 D^{1/2} + 2\beta_3 - \frac{\beta_3^2 - 4\pi^2}{2\pi^2} D^{-1/2} \right) + O(D^{-1}),$$

where the third term is negative, which suggests the following inequality:

$$(3.47) \quad \|u\|_\infty^2 \leq \frac{1}{2\pi} \|\nabla u\|_{L_2(\mathbb{T}^3)} \|\Delta u\|_{L_2(\mathbb{T}^3)} - \frac{\beta_3}{4\pi^3} \|\nabla u\|_{L_2(\mathbb{T}^3)}^2,$$

where all constants are sharp. However, up to the moment, we have checked this inequality only for large D . The next lemma shows that it holds for all $D \geq 1$.

Lemma 3.10. *The inequality (3.47) holds for every $u \in H^2(\mathbb{T}^3)$ with zero mean.*

Proof. We only need to check that the inequality

$$(3.48) \quad \mathbb{V}(D) \leq \frac{1}{8\pi^3} \left(4\pi^2 D^{1/2} + 2\beta_3 \right)$$

holds for all $D \geq 1$. To this end, we will again use the variational representation (3.27) where we put $\lambda = D - \frac{1}{2}$ and inequality (3.45). Singling out the term $4\pi^2 D^{1/2} + 2\beta_3$ this gives

$$(3.49) \quad 8\pi^3 \mathbb{V}(D) \leq \left(1 + \frac{D}{D-1/2} \right) \left(2\pi^2 \sqrt{D-1/2} + \beta_3 + \frac{1}{D-1/2} \right) = \\ = 4\pi^2 D^{1/2} + 2\beta_3 + \frac{1}{2D-1} \left(\pi^2 \frac{\sqrt{4D-2}}{\sqrt{4D(4D-2)} + 4D-1} + \beta_3 + 4 + \frac{2}{2D-1} \right),$$

and we only need to check that the last term in the right-hand side is always negative. Indeed,

$$\pi^2 \frac{\sqrt{4D-2}}{\sqrt{4D(4D-2)} + 4D-1} < \pi^2 \frac{\sqrt{4D-2}}{\sqrt{4D(4D-2)} + 4D-2} = \pi^2 \frac{1}{\sqrt{4D} + \sqrt{4D-2}} \leq \frac{\pi^2}{2 + \sqrt{2}}$$

if $D \geq 1$ and, analogously, $\frac{2}{2D-1} \leq 2$. Then

$$\frac{\pi^2}{2 + \sqrt{2}} + \beta_3 + 6 = -0.022892 < 0$$

and the lemma is proved. \square

3.2.1.3. Computing the integration constants.

As we have seen above, in the case $l > 0$ the Poisson summation formula allows to find the asymptotic expansions of the Green's function $G(\lambda)$ only up to some integration constants and the direct computation of that constants is a non-trivial task since the series (3.12) converge not sufficiently fast, especially for big λ . Thus, it looks reasonable to find better (e.g., exponentially) convergent series for computing them. In the present section, we give an explicit formula for the sum (3.12) in the particular case $n = 3$, $m = 2$, $l = 1$ considered above in terms of the integrals of the so-called Jacobi theta functions and using the known relations for the theta functions, we find the formula for the integration constant β_3 through the very fast convergent and convenient for computations series. Note also that, although we restrict ourselves to consider only that case, the presented method has a general nature and is applicable for computing other integration constants including the case of anisotropic tori, etc.

We consider the Jacobi theta function (see, for instance, [9])

$$(3.50) \quad \theta_3(q) = \sum_{k \in \mathbb{Z}} q^{k^2}.$$

Then, the following identity holds which is crucial in what follows:

$$(3.51) \quad \theta_3(e^{-\pi t}) = \frac{1}{\sqrt{t}} \cdot \theta_3(e^{-\pi t^{-1}}),$$

and which follows from the Poisson summation formula (3.13) with $n = 1$, $f(x) = e^{-x^2/2}$, $\widehat{f}(\xi) = e^{-\xi^2/2}$, and $\mu = \sqrt{\frac{t}{2\pi}}$.

Using also the obvious relation

$$\frac{1}{k(1+\mu k)} = \int_0^\infty (1 - e^{-t/\mu}) e^{-kt} dt,$$

we transform (3.41) as follows

$$8\pi^3 \tilde{G}(\mu) = \int_0^\infty (1 - e^{-t/\mu}) \sum_{k \in \mathbb{Z}_0^3} [e^{-t}]^{k_1^2} [e^{-t}]^{k_2^2} [e^{-t}]^{k_3^2} dt = \int_0^\infty (1 - e^{-t/\mu}) ([\theta_3(e^{-t})]^3 - 1) dt.$$

Splitting the interval of integration $\mathbb{R}_+ = [0, 1] \cup [1, \infty)$ and using (3.51), we arrive at

$$\begin{aligned} (3.52) \quad 8\pi^3 \tilde{G}(\mu) &= \int_1^\infty (1 - e^{-t/\mu}) ([\theta_3(e^{-t})]^3 - 1) dt + \\ &\quad + \int_0^1 (1 - e^{-t/\mu}) \left(\pi^{3/2} t^{-3/2} \theta_3(e^{-\pi^2 t^{-1}})^3 - 1 \right) dt \\ &= \int_1^\infty (1 - e^{-t/\mu}) ([\theta_3(e^{-t})]^3 - 1) dt + \pi^{3/2} \int_1^\infty (1 - e^{-1/(t\mu)}) t^{-1/2} \left(\theta_3(e^{-\pi^2 t})^3 - 1 \right) dt - \\ &\quad - \int_0^1 (1 - e^{-t/\mu}) dt + \pi^{3/2} \int_0^1 t^{-3/2} (1 - e^{-t/\mu}) dt. \end{aligned}$$

Now it is not difficult to find the asymptotic expansions for $\tilde{G}(\mu)$ as $\mu \rightarrow 0+$. Indeed, as elementary calculations show,

$$\begin{aligned} (3.53) \quad \int_0^1 t^{-3/2} (1 - e^{-t/\mu}) dt &= \mu^{-1/2} \left(\int_0^\infty t^{-3/2} (1 - e^{-t}) dt - \int_{1/\mu}^\infty t^{-3/2} (1 - e^{-t}) dt \right) = \\ &= \mu^{-1/2} (2\sqrt{\pi} - \int_{1/\mu}^\infty t^{-3/2} dt + O(e^{-1/\mu})) = 2\sqrt{\pi} \mu^{-1/2} - 2 + o_\mu(1). \end{aligned}$$

Using now that $\theta_3(x) - 1 = O(x)$ as $x \rightarrow 0$ we can pass to the limit $\mu \rightarrow 0$ in the integrals in (3.52) containing θ_3 and comparing with (3.44), we see that

$$(3.54) \quad \beta_3 = -1 - 2\pi^{3/2} + \int_1^\infty (\theta_3(e^{-t})^3 - 1) dt + \pi^{3/2} \int_1^\infty t^{-1/2} (\theta_3(e^{-\pi^2 t})^3 - 1) dt.$$

We do not know whether or not the integral in (3.54) can be computed in closed form, however, it is convenient for high precision numerical computation of the constant β_3 . Indeed, expanding back the Jacobi function in Taylor series and using that

$$\int_1^\infty t^{-1/2} e^{-kt} dt = \sqrt{\pi} \frac{\operatorname{erfc}(\sqrt{k})}{\sqrt{k}},$$

where $\operatorname{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$, we end up with

$$(3.55) \quad \beta_3 = -1 - 2\pi^{3/2} + \sum_{k \in \mathbb{Z}_0^3} \left(\frac{e^{-|k|^2}}{|k|^2} + \pi \frac{\operatorname{erfc}(\pi|k|)}{|k|} \right).$$

We see that the rate of convergence of the series is super-exponential and using Maple to compute it, we get the desired value

$$\beta_3 = -8.91363291758515127$$

which has been used in the previous section.

3.2.2. *Inequalities on spheres.* We recall the basic facts concerning the spectrum of the Laplace-Beltrami operator on the $(d-1)$ -dimensional sphere \mathbb{S}^{d-1} :

$$-\Delta Y_n^k = \Lambda_n Y_n^k, \quad k = 1, \dots, k_d(n), \quad n = 1, 2, \dots$$

Here the Y_n^k are the orthonormal spherical harmonics. Each eigenvalue

$$\Lambda_n = n(n+d-2)$$

has multiplicity

$$k_d(n) = \frac{2n+d-2}{n} \binom{n+d-3}{n-1}.$$

In particular, for $d = 3, 4$ we have

$$(3.56) \quad \begin{aligned} \mathbb{S}^2 : \quad \Lambda_n &= n(n+1), \quad k_3(n) = 2n+1, \\ \mathbb{S}^3 : \quad \Lambda_n &= n(n+2), \quad k_4(n) = (n+1)^2. \end{aligned}$$

The following identity is essential [22]: for any $\xi \in \mathbb{S}^{d-1}$

$$(3.57) \quad \sum_{l=1}^{k_d(n)} Y_n^l(\xi)^2 = \frac{k_d(n)}{\sigma(d)},$$

where $\sigma(d) = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of S^{d-1} .

Finally, since the kernel of Δ is the one-dimensional subspace of constants, throughout below we assume orthogonality to constants:

$$\bar{H}^s(\mathbb{S}^{d-1}) = \{\varphi \in H^s(\mathbb{S}^{d-1}), (\varphi, 1) = 0\}.$$

3.2.2.1. *Inequalities on spheres: \mathbb{S}^2 .* We consider below applications of the general theory to inequalities on the 2D sphere with $A = (-\Delta)^m$, $m > 1/2$, and $B = I$, where Δ is the Laplace-Beltrami operator. The first positive eigenvalue of $-\Delta$ is 2 and in accordance with (2.13) we consider the operator

$$\mathbb{A}(\lambda) = (-\Delta)^m + \lambda I, \quad \lambda > -2^m.$$

Its Green's function $G_\lambda(x, \xi)$ is

$$G_\lambda(x, \xi) = \sum_{n=1}^{\infty} \sum_{k=1}^{2n+1} \frac{Y_n^k(\xi) Y_n^k(x)}{(n(n+1))^m + \lambda}, \quad x, \xi \in \mathbb{S}^2,$$

and thanks to (3.57) the function $G_\lambda(\xi, \xi)$ is independent of ξ and is given by

$$(3.58) \quad G(\lambda) = G_\lambda(\xi, \xi) = \sum_{n=1}^{\infty} \sum_{k=1}^{2n+1} \frac{Y_n^k(\xi)^2}{(n(n+1))^m + \lambda} = \frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{2n+1}{(n(n+1))^m + \lambda}.$$

Setting $\mu := \lambda^{-1/m}$ we have

$$G(\lambda) = \frac{1}{4\pi} \mu^m \sum_{n=1}^{\infty} (2n+1) \varphi(\mu n(n+1)),$$

where

$$\varphi(x) = \frac{1}{x^m + 1}.$$

Since $\varphi(0) = 1$ and $\varphi'(0) = 0$, Lemma 3.14 below gives

$$(3.59) \quad G(\lambda) = \frac{1}{4\pi} \left[\lambda^{-\theta} K - \frac{2}{3} \lambda^{-1} + 0 \cdot \lambda^{-(2-\theta)} \right] + O(\lambda^{-(3-2\theta)}),$$

where $\theta = (m-1)/m$ and

$$K = \int_0^\infty \varphi(x) dx = \frac{(1-\theta)\pi}{\sin \theta\pi}.$$

In other words, we have shown that for $f_\xi(\lambda) = G(\lambda)$ the asymptotic expansion (2.41) holds with the following g_1, g_2, g_3 independent of $\xi \in \mathbb{S}^2$:

$$(3.60) \quad g_1 = \frac{K}{4\pi} = \frac{1-\theta}{4\sin \theta\pi}, \quad g_2 = -\frac{1}{6\pi}, \quad g_3 = 0.$$

Next, differentiating (3.58) and again using Lemma 3.14 we obtain the asymptotic expansion for $g_\xi(\lambda) = g(\lambda)$:

$$\begin{aligned} g(\lambda) = -f'(\lambda) = -G'(\lambda) &= \frac{1}{4\pi} \sum_{n=1}^\infty \frac{2n+1}{((n(n+1))^m + \lambda)^2} = \frac{1}{4\pi} \mu^{2m} \sum_{n=1}^\infty \frac{2n+1}{((\mu n(n+1))^m + 1)^2} = \\ &= \frac{1}{4\pi} \left[\lambda^{-(1-\theta)} \theta K - \frac{2}{3} \lambda^{-2} + 0 \cdot \lambda^{-(3-\theta)} \right] + O(\lambda^{-(4-\theta)}), \end{aligned}$$

where we used

$$\int_0^\infty \varphi(x)^2 dx = \int_0^\infty \frac{dx}{(x^m + 1)^2} = \theta K.$$

This justifies differentiation of the asymptotic formula for $f(\lambda)$, as required in Proposition 2.6. Applying it we obtain as a result the asymptotics of the function $\mathbb{V}(D)$ on \mathbb{S}^2 .

Theorem 3.11. *The function $\mathbb{V}(D)$ solving on \mathbb{S}^2 the extremal problem*

$$(3.61) \quad \mathbb{V}(\xi, D) := \sup \left\{ |u(\xi)|^2 : u \in \bar{H}^m(\mathbb{S}^2), \quad \|u\|^2 = 1, \quad \|(-\Delta)^{m/2} u\|^2 = D \right\}$$

is independent of $\xi \in \mathbb{S}^2$ and has the following asymptotic behavior as $D \rightarrow \infty$:

$$(3.62) \quad \mathbb{V}(D) = \frac{1}{4\sin \theta\pi} \left(\frac{1-\theta}{\theta} \right)^\theta D^{1-\theta} - \frac{1}{6\pi\theta} - \frac{\sin \theta\pi}{18\pi^2} \frac{(1-\theta)^{1-\theta}}{\theta^{3-\theta}} D^{\theta-1} + O(D^{-(2-2\theta)}),$$

where $\theta = (m-1)/m$.

We now apply the asymptotics of $\mathbb{V}(D)$ obtained above to multiplicative inequalities with remainder terms on \mathbb{S}^2 . By Theorem 2.5 the sharp constant $K = K_\theta$ in the classical multiplicative inequality

$$(3.63) \quad u(\xi)^2 \leq \|u\|_\infty^2 \leq K_\theta \|u\|^{2\theta} \|(-\Delta)^m u\|^{2(1-\theta)}$$

is given by

$$(3.64) \quad K_\theta = \frac{1}{\theta^\theta (1-\theta)^{1-\theta}} \sup_{\lambda \geq 0} \lambda^\theta G(\lambda) = \frac{1}{\theta^\theta (1-\theta)^{1-\theta}} \sup_{\lambda \geq 0} \frac{1}{4\pi} \lambda^\theta \sum_{n=1}^\infty \frac{2n+1}{(n(n+1))^m + \lambda}.$$

We note that (3.64) was obtained in [12] by a somewhat similar but less general argument than the one used in Theorem 2.5. It was also shown there that for (integer) m , $2 \leq m \leq 7$, we in fact have that the supremum is attained at infinity

$$(3.65) \quad \sup_{\lambda \geq 0} \lambda^\theta G(\lambda) = \lim_{\lambda \rightarrow \infty} \lambda^\theta G(\lambda) = \int_0^\infty \frac{dx}{1+x^m},$$

which gives

$$K_\theta = \frac{1}{4\sin \theta\pi} \left(\frac{1-\theta}{\theta} \right)^\theta$$

and, equivalently,

$$(3.66) \quad \mathbb{V}(D) = \mathbb{V}_m(D) < \frac{1}{4 \sin \theta \pi} \left(\frac{1-\theta}{\theta} \right)^\theta D^{1-\theta}$$

for $2 \leq m \leq 7$. However, for larger m 's the supremum in (3.65) is attained at a finite point $\lambda_* < \infty$. An explanation of this phenomenon for the torus has been given in [2]; below we consider the case of the sphere \mathbb{S}^2 .

Lemma 3.12. *For all sufficiently large m the function $h(\lambda) := \lambda^\theta G(\lambda)$ attains a global maximum at a finite point λ_* and $h(\lambda_*) > h(\infty)$.*

Proof. Setting in (3.64) $\lambda = \nu^{2m}$ we see that up to a constant factor, $h(\lambda)$ is equal to

$$H(\nu) = \nu^{2m-2} \sum_{n=1}^{\infty} \frac{2n+1}{\nu^{2m} + (n(n+1))^m}.$$

We consider the following partitioning of the half-line $x \geq 0$ by the points

$$a_n = a_n(\nu) = \frac{(n-1)n}{\nu^2}, \quad n = 1, \dots$$

Then a direct inspection shows that

$$(3.67) \quad H(\nu) = \frac{1}{2} \varphi(a_2)(a_2 - a_1) + \sum_{n=2}^{\infty} \frac{\varphi(a_n) + \varphi(a_{n+1})}{2} (a_{n+1} - a_n),$$

where $\varphi(x) = 1/(1+x^m)$, which looks like a step function for large m : $\varphi(x) \approx 1$ for $x < 1$ and $\varphi(x) \approx 0$ for $x > 1$. In view of Lemma 3.14 below we have

$$H(\infty) := \lim_{\nu \rightarrow \infty} H(\nu) = \int_0^\infty \varphi(x) dx = \frac{\pi/m}{\sin \pi/m} = 1 + o_{m \rightarrow \infty}(1).$$

We fix a large m , and set, say, $\nu = \nu_0 = \sqrt{2} + 1/100$. Then

$$\begin{aligned} a_1(\nu_0) &= 0, \quad a_2(\nu_0) < 1 (= 0.986), \quad a_3(\nu_0) \approx 3 (= 2.958); \\ a_2(\nu_0) - a_1(\nu_0) &= 0.986; \quad a_3(\nu_0) - a_2(\nu_0) = 1.972. \end{aligned}$$

Since $\varphi(a_2) = 1 + o_{m \rightarrow \infty}(1)$, $\varphi(a_3) = o_{m \rightarrow \infty}(1)$, and the sum from $n = 3$ to ∞ in (3.67) gives a contribution of the order $o_{m \rightarrow \infty}(1)$, it follows that

$$H(\nu_0) = \frac{0.986}{2} + \frac{1.972}{2} + o_{m \rightarrow \infty}(1) = 1.479 + o_{m \rightarrow \infty}(1).$$

Hence $H(\nu_0) > H(\infty)$, and the proof is complete. \square

This argument also explains the initial oscillatory behavior of $H(\nu)$ when the corresponding node $a_n(\nu)$ hits 1 with the increase of ν . Figure 2 shows the monotone behavior of $H(\nu)$ for $m = 2$ and the initial oscillations of $H(\nu)$ for $m = 10$. Observe that the first (and global) maximum in the second case is located near $\nu = \sqrt{2}$.

Turning to the inequalities with remainder terms we note that the third term in (3.62) is negative, therefore the improved inequality

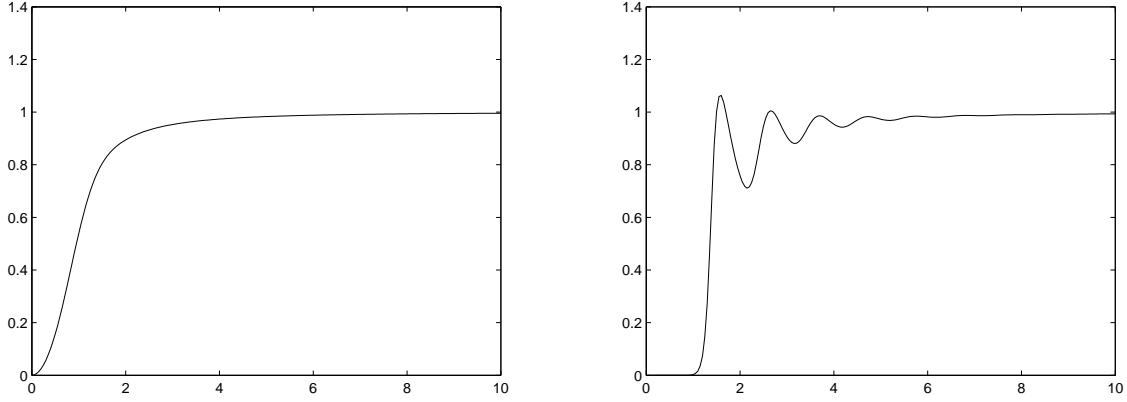
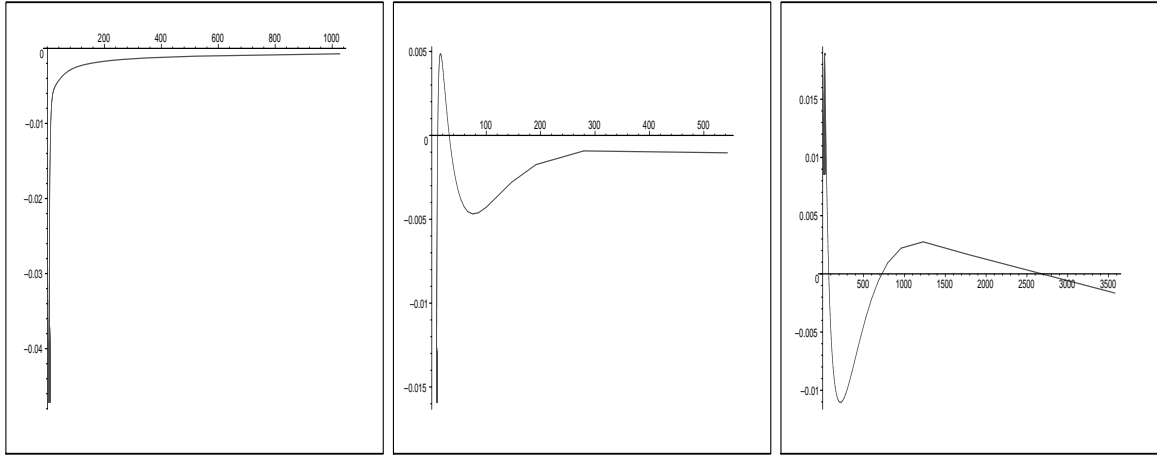
$$(3.68) \quad \mathbb{V}(D) < \frac{1}{4 \sin \theta \pi} \left(\frac{1-\theta}{\theta} \right)^\theta D^{1-\theta} - \frac{1}{6\pi\theta} =: \overline{\mathbb{V}}(D)$$

holds for all $D \geq D_0$, where D_0 is sufficiently large. On the finite interval $[2^m, D_0]$ computer calculations are reliable; their results are shown in Fig.3.

In particular, for $m = 2$

$$\mathbb{V}(D) < \overline{\mathbb{V}}(D) \quad \text{for all } D,$$

which proves the following result.

FIGURE 2. Graphs of the normalized function $H(\nu)$ for $m = 2$ and $m = 10$ FIGURE 3. Graph of the function $\mathbb{V}(D) - \overline{\mathbb{V}}(D)$ for $m = 2, 3, 4$.

Theorem 3.13. *If $u \in \bar{H}^2(\mathbb{S}^2)$, then*

$$(3.69) \quad \|u\|_\infty^2 \leq \frac{1}{4} \|u\| \|\Delta u\| - \frac{1}{3\pi} \|u\|^2,$$

where both constants are sharp and no extremal functions exist.

For $m = 3$ and $m = 4$ the function $\mathbb{V}_m(\xi, D) - \overline{\mathbb{V}}_m(\xi, D)$ attains a global maximum $0.00486\dots$ at $D = 15.8\dots$ for $m = 3$ and a global maximum $0.0189\dots$ at $D = 22.4\dots$ for $m = 4$, respectively. Accordingly, we have

$$(3.70) \quad \|u\|_\infty^2 \leq \frac{1}{4 \sin \theta \pi} \left(\frac{1 - \theta}{\theta} \right)^\theta \|u\|^{2\theta} \|(-\Delta)^{m/2} u\|^{2(1-\theta)} - \frac{\varepsilon_m}{6\pi\theta} \|u\|^2,$$

where $m = 3, 4$, $\theta = (m - 1)/m$ and $\varepsilon_3 = 0.938\dots$, $\varepsilon_4 = 0.821\dots$.

We now prove the asymptotic formula for functions defined by the following series (3.71) that we have been systematically using above. Let $F(\mu)$ be defined as follows

$$(3.71) \quad F(\mu) := \sum_{n=1}^{\infty} (2n + 1) f(\mu n(n + 1)),$$

where f is sufficiently smooth and sufficiently fast decays at infinity. We need to find the asymptotics of $F(\mu)$ as $\mu \rightarrow 0$.

Lemma 3.14. *The following asymptotic expansion holds as $\mu \rightarrow 0$:*

$$(3.72) \quad F(\mu) = \frac{1}{\mu} \int_0^\infty f(x)dx - \frac{2}{3}f(0) - \frac{1}{15}\mu f'(0) + O(\mu^2).$$

Proof. We set

$$R(x) = (2x+1)f(\mu x(x+1))$$

and observe that

$$(3.73) \quad \int_0^\infty R(x)dx = \frac{1}{\mu} \int_0^\infty f(\mu x(x+1))d(\mu x(x+1)) = \frac{1}{\mu} \int_0^\infty f(x)dx.$$

We calculate the derivatives of R up the order 5 at $x = 0$:

$$(3.74) \quad \begin{aligned} R(0) &= f(0); \\ R'(0) &= 2f(0) + \mu f'(0); \\ R''(0) &= 6\mu f'(0) + \mu^2 f''(0); \\ R'''(0) &= 12\mu f'(0) + 12\mu^2 f''(0) + \mu^3 f'''(0); \\ R^{(4)}(0) &= 60\mu^2 f''(0) + 20\mu^3 f'''(0) + \mu^4 f^{(4)}(0); \\ R^{(5)}(0) &= 120\mu^2 f''(0) + 180\mu^3 f'''(0) + 30\mu^4 f^{(4)}(0) + \mu^5 f^{(5)}(0), \end{aligned}$$

and, in addition,

$$(3.75) \quad \begin{aligned} R^{(6)}(x) &= 840\mu^3(2x+1)f^{(3)}(\mu x(x+1)) + 420\mu^4(2x+1)^3 f^{(4)}(\mu x(x+1)) + \\ &\quad + 42\mu^5(2x+1)^5 f^{(5)}(\mu x(x+1)) + \mu^6(2x+1)^7 f^{(6)}(\mu x(x+1)). \end{aligned}$$

Next we use the Euler–Maclaurin formula (see, for instance, [16])

$$(3.76) \quad \sum_{n=0}^\infty R(n) = \int_0^\infty R(x)dx + \frac{1}{2}R(0) - \sum_{i=2}^k \frac{B_i}{i!} R^{(i-1)}(0) - \int_0^\infty \frac{B_k(x)}{k!} R^{(k)}(x)dx,$$

where the B_k 's are the Bernoulli numbers: $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, $B_5 = 0$, $B_6 = \frac{1}{42}$, \dots , and the $B_k(x)$'s are the periodic Bernoulli polynomials. Using (3.76) with $k = 6$ and taking into account (3.73) and (3.74) we obtain

$$\begin{aligned} F(\mu) &= -R(0) + \sum_{n=0}^\infty R(n) = \frac{1}{\mu} \int_0^\infty f(x)dx - \frac{1}{2}f(0) - \frac{1}{12}(2f(0) + \mu f'(0)) + \\ &\quad + \frac{1}{720}(12\mu f'(0) + 12\mu^2 f''(0) + \mu^3 f'''(0)) - \\ &\quad - \frac{1}{42 \cdot 720}(120\mu^2 f''(0) + 180\mu^3 f'''(0) + 30\mu^4 f^{(4)}(0) + \mu^5 f^{(5)}(0)) - \int_0^\infty \frac{B_6(x)}{6!} R^{(6)}(x)dx. \end{aligned}$$

This gives (3.72) provided that the remainder integral term is of the order $O(\mu^2)$. The periodic Bernoulli polynomials are clearly bounded on $(0, \infty)$. Therefore the contribution of each term in (3.75) is of the order μ^2 . For example, the last term is of the order

$$\mu^6 \int_0^\infty x^7 g(\mu x^2)dx = \frac{1}{2}\mu^2 \int_0^\infty y^3 g(y)dy.$$

The three remaining terms in (3.75) are treated similarly. □

3.2.2.2. Inequalities on spheres: \mathbb{S}^3 . We consider on the 3D sphere \mathbb{S}^3 only one example with $l = 1$ and $m = 2$, so that $\theta = 1/2$. We set $A = (-\Delta)^2$, $B = -\Delta$, and let

$$\mathbb{A}(\lambda) = (-\Delta)^2 - \lambda\Delta.$$

The Green's function of $\mathbb{A}(\lambda)$ is

$$G_\lambda(x, \xi) = \sum_{n=1}^{\infty} \sum_{k=1}^{(n+1)^2} \frac{Y_n^k(\xi) Y_n^k(x)}{n(n+2)(n(n+2) + \lambda)}$$

and again using (3.57) we see that $G_\lambda(\xi, \xi)$ is independent of ξ :

$$(3.77) \quad G(\lambda) = G_\lambda(\xi, \xi) = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)(n(n+2) + \lambda)}, \quad \text{for any } \xi \in \mathbb{S}^3.$$

We do not need an analogue of Lemma 3.14 here since the series in (3.77) can be summed explicitly [12]:

$$(3.78) \quad G(\lambda) = \frac{1}{2\pi^2} \left[\frac{\sqrt{\lambda-1}}{\lambda} \frac{\pi}{2} \coth \pi \sqrt{\lambda-1} - \frac{\lambda-1}{\lambda^2} + \frac{1}{4\lambda} \right],$$

and, in addition,

$$(3.79) \quad \sup_{\lambda \geq 0} \lambda^\theta G(\lambda) = \lim_{\lambda \rightarrow \infty} \lambda^\theta G(\lambda) = \frac{1}{4\pi},$$

which gives in view of Theorem 2.5 the following sharp inequality on \mathbb{S}^3 [12]

$$(3.80) \quad \|u\|_\infty^2 \leq \frac{1}{2\pi} \|\nabla u\| \|\Delta u\|.$$

To obtain power expansion of $G(\lambda)$ we can replace $\coth \pi \sqrt{\lambda-1}$ by 1 which gives :

$$G(\lambda) = \frac{1}{2\pi^2} \left[\frac{\pi}{2} \lambda^{-1/2} - \frac{3}{4} \lambda^{-1} - \frac{\pi}{4} \lambda^{-3/2} + O(\lambda^{-2}) \right],$$

so that (2.41) holds with:

$$(3.81) \quad g_1 = \frac{1}{4\pi}, \quad g_2 = -\frac{3}{8\pi^2}, \quad g_3 = -\frac{1}{8\pi}.$$

Using Proposition 2.6 we obtain the following result.

Theorem 3.15. *The solution $\mathbb{V}(D)$ of the extremal problem*

$$(3.82) \quad \mathbb{V}(\xi, D) := \sup \left\{ |u(\xi)|^2 : u \in \bar{H}^2(\mathbb{S}^3), \quad \|\nabla u\|^2 = 1, \quad \|\Delta u\|^2 = D \right\}$$

is independent of $\xi \in \mathbb{S}^3$ and has the following asymptotic behavior as $D \rightarrow \infty$:

$$(3.83) \quad \mathbb{V}(D) = \frac{1}{2\pi} D^{1/2} - \frac{3}{4\pi^2} - \left(\frac{9+4\pi^2}{16\pi^3} \right) D^{-1/2} + O(D^{-1}).$$

The third term in (3.83) is negative, therefore

$$(3.84) \quad \mathbb{V}(D) < \frac{1}{2\pi} \sqrt{D} - \frac{3}{4\pi^2}$$

for all $D \geq D_0$, where D_0 is sufficiently large. However, similarly to Theorem 3.7 and Lemma 3.10, taking the advantage of the explicit formula (3.78), we have the following result.

Lemma 3.16. *Inequality (3.84) holds for all $D \geq \sqrt{3}$.*

Proof. In view of Theorem 2.8, for $D \geq \sqrt{3}$ (the first eigenvalue of $(-\Delta)^{1/2}$ on \mathbb{S}^3 is $\sqrt{3}$)

$$(3.85) \quad \mathbb{V}(D) = \min_{\lambda \geq -\sqrt{3}} \{(\lambda + D)G(\lambda)\} \leq \{(\lambda + D)G(\lambda)\}_{\lambda=D} = 2DG(D).$$

Hence, using (3.78), the inequality $\sqrt{D-1} \leq \sqrt{D} - 1/(2\sqrt{D}) - 1/(8D^{3/2})$, and replacing $\coth \alpha$ in the negative terms below by 1, we obtain

$$\begin{aligned} & \mathbb{V}(D) - \frac{1}{2\pi}\sqrt{D} + \frac{3}{4\pi^2} \leq 2DG(D) - \frac{1}{2\pi}\sqrt{D} + \frac{3}{4\pi^2} = \\ & \frac{2D}{2\pi^2} \left(\frac{\sqrt{D-1}}{D} \frac{\pi}{2} \coth \pi\sqrt{D-1} - \frac{D-1}{D^2} + \frac{1}{4D} \right) - \frac{1}{2\pi}\sqrt{D} + \frac{3}{4\pi^2} = \\ & \frac{1}{2\pi} \left(\sqrt{D} \left(\coth \pi\sqrt{D-1} - 1 \right) - \frac{1}{2\sqrt{D}} - \frac{1}{8D^{3/2}} + \frac{2}{\pi D} \right) = \\ & = \frac{1}{2\pi D^{3/2}} \left(D^2 \left(\coth \pi\sqrt{D-1} - 1 \right) - \frac{D}{2} - \frac{1}{8} + \frac{2\sqrt{D}}{\pi} \right) = \frac{1}{2\pi D^{3/2}} (F_1(D) - F_2(D)) < 0, \end{aligned}$$

since $F_1(D) = D^2 (\coth \pi\sqrt{D-1} - 1)$ is decreasing for $D \geq \sqrt{3}$ and $F_1(D) \leq F_1(\sqrt{3}) = 3 \coth(\pi\sqrt{\sqrt{3}-1} - 1) = 0.0278\dots$, while $F_2(D) = D/2 + 1/8 - 2\sqrt{D}/\pi$ is increasing and $F_2(D) \geq F_2(\sqrt{3}) = \sqrt{3}/2 + 1/8 - 2\sqrt{3}/\pi = 0.153\dots$. The proof is complete. \square

As an immediate corollary we obtain inequality (3.80) in the following refined form.

Theorem 3.17. *If $u \in \bar{H}^2(\mathbb{S}^3)$, then*

$$(3.86) \quad \|u\|_\infty^2 \leq \frac{1}{2\pi} \|\nabla u\| \|\Delta u\| - \frac{3}{4\pi^2} \|\nabla u\|^2,$$

where both constants are sharp and no extremal functions exist.

Remark 3.18. The coefficient of the leading term in the asymptotic expansions (3.62), (3.83) coincides with the corresponding constant in \mathbb{R}^n .

3.3. Manifolds with boundary.

3.3.1. First-order inequality on the half-line. We consider a couple of inequalities on the half-line $\mathbb{R}_+ = (0, \infty)$. On the whole \mathbb{R} we have

$$(3.87) \quad \|u\|_\infty^2 \leq \|u\| \|u'\|,$$

where $u \in H^1(\mathbb{R})$ and where the constant 1 is sharp and there exists a unique extremal function $u_*(x) = e^{-|x|}$. Using the extension by zero we see that inequality (3.87) still holds on \mathbb{R}_+ , and there are no extremal functions since $u_*(x) > 0$. We can obtain the following refined form of this inequality

$$(3.88) \quad u(\xi)^2 \leq \|u\| \|u'\| (1 - e^{-2\xi}), \quad u \in H_0^1(\mathbb{R}_+)$$

from the general theory developed above. In fact, the Green's function of the operator

$$\mathbb{A}u = -u'' + u, \quad u(0) = u(\infty) = 0$$

is

$$G(x, \xi) = \begin{cases} e^{-\xi} \sinh x, & x < \xi, \\ e^{-x} \sinh \xi, & x \geq \xi. \end{cases}$$

As in Lemma 2.1 we have

$$u(\xi)^2 \leq (\|u'\|^2 + \|u\|^2) G(\xi, \xi) = (\|u'\|^2 + \|u\|^2) \frac{1 - e^{-2\xi}}{2},$$

and (3.88) follows by the standard scaling argument.

3.3.2. Inequality for Bessel operator on the half-line. Here we consider a one-dimensional inequality on the half-line \mathbb{R}^+ with $Au = -u'' - u/4x^2$ and $Bu = u$. Although the corresponding sharp constant was found in [17] in connection with Lieb–Thirring inequalities for radial potentials (see also [8]) we include this example to illustrate our general approach. We first observe that in view of the one-dimensional Hardy inequality

$$\int_0^\infty \frac{u(x)^2}{x^2} dx \leq 4 \int_0^\infty (u'(x))^2 dx,$$

the operator A is non-negative. In accordance with the general theory developed in section 2 in order to find the sharp constant $C = C(\xi)$ in the inequality

$$(3.89) \quad |u(\xi)|^2 \leq C(\xi) \left(\int_0^\infty \left((u'(x))^2 - \frac{u(x)^2}{4x^2} \right) dx \right)^{1/2} \|u\|,$$

we need to write down the Green's function of the following operator

$$\mathbb{A}(\lambda)u = -u'' - u/4x^2 + \lambda u$$

with Dirichlet boundary conditions $u(0) = u(\infty) = 0$. We have

$$(3.90) \quad G_\lambda(x, \xi) = \begin{cases} \sqrt{x\xi} K_0(\sqrt{\lambda\xi}) I_0(\sqrt{\lambda}x), & x < \xi, \\ \sqrt{x\xi} K_0(\sqrt{\lambda}x) I_0(\sqrt{\lambda\xi}), & x > \xi, \end{cases}$$

where K_0 and I_0 are the modified Bessel functions of zeroth order. In fact, the functions $\sqrt{x}K_0(\sqrt{\lambda}x)$ and $\sqrt{x}I_0(\sqrt{\lambda}x)$ both satisfy the homogeneous equation, $I_0(0) = 0$, $K_0(\infty) = 0$, and the jump condition $G(\xi + 0, \xi) - G(\xi - 0, \xi) = 1$ is satisfied in view of the Wronski identity $I_0(x)K_0'(x) - I_0'(x)K_0(x) = -1/x$. Therefore by Theorem 2.5, the constant $C(\xi)$ is, in fact, independent of ξ and is given by

$$\begin{aligned} C(\xi) &= 2 \sup_{\lambda \geq 0} \sqrt{\lambda} G_\lambda(\xi, \xi) = 2 \sup_{\lambda \geq 0} \sqrt{\lambda\xi} K_0(\sqrt{\lambda\xi}) I_0(\sqrt{\lambda\xi}) = \\ &= 2 \sup_{r \geq 0} r K_0(r) I_0(r) = 2r K_0(r) I_0(r)|_{r=r_*=1.075\dots} = 2 \cdot 0.533\dots = 1.06\dots =: C_*. \end{aligned}$$

Furthermore, for every fixed $\xi > 0$ inequality (3.89) turns into equality for $C(\xi) = C_*$ and $u(x) = G_{\lambda_*}(x, \xi)$, where $\lambda_* = \lambda_*(\xi) = r_*^2/\xi^2$. We also observe that $I_0(x) \sim \sqrt{\frac{1}{2\pi x}} e^x$ and $K_0(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}$ as $x \rightarrow \infty$, and hence

$$\lim_{r \rightarrow \infty} r K_0(r) I_0(r) = \frac{1}{2}.$$

3.3.3. First-order inequality on interval. In this section we consider the correction term for the interpolation inequality

$$(3.91) \quad \|u\|_\infty^2 \leq 1 \cdot \|u\| \|u'\|, \quad u \in H_0^1(0, L)$$

on an interval $(0, L)$ with zero boundary conditions. The constant 1 is sharp, since it is sharp for the inequality (3.87) on \mathbb{R} and we can use extension by zero. The unique extremal function in (3.87) does not vanish, therefore there are no extremal functions in (3.91), and one might expect that an inequality similar to (3.22) holds in the case of a finite interval:

$$(3.92) \quad \|u\|_\infty^2 \leq \|u\| \|u'\| - c_L \|u\|^2, \quad u \in H_0^1(0, L).$$

In fact, $c_L = 0$ as we now show. First, by scaling, if (3.92) holds, then $c_L = c/L$ where c is an absolute constant. Next, we consider the truncated extremal function

$$\varphi_a(x) = \begin{cases} e^{-x}, & 0 \leq x \leq a; \\ e^{-a}(a+1-x), & a \leq x \leq a+1; \\ 0, & x \geq 0, \end{cases}$$

and set $\varphi_a(-x) = \varphi_a(x)$. Then

$$\|\varphi_a\|_\infty = 1, \quad \|\varphi_a\|^2 = 1 - \frac{1}{3}e^{-2a}, \quad \|\varphi'_a\|^2 = 1 + e^{-2a}.$$

Substituting this into (3.92) and letting $a \rightarrow \infty$ we obtain that (3.92) can only hold with $c = 0$ as claimed.

However, the correction term exists but is exponentially small. More precisely, we have the following result.

Theorem 3.19. *Let $u \in H_0^1(0, L)$. Then*

$$(3.93) \quad \|u\|_\infty^2 \leq \|u\| \|u'\| \left(1 - 2e^{-\frac{L\|u'\|}{\|u\|}}\right).$$

The coefficients of the two terms on the right-hand side are sharp and no extremal functions exist.

Proof. Without loss of generality we set $L = 1$. The Green's function of the boundary value problem

$$-y'' + \lambda y = \delta(x, \xi), \quad y(0) = y(1) = 0$$

is

$$(3.94) \quad G_\lambda(x, \xi) = \frac{1}{\lambda^{1/2} \sinh \lambda^{1/2}} \begin{cases} \sinh \lambda^{1/2} x \sinh \lambda^{1/2} (1 - \xi), & 0 \leq x \leq \xi; \\ \sinh \lambda^{1/2} \xi \sinh \lambda^{1/2} (1 - x), & \xi \leq x \leq 1. \end{cases}$$

In addition, $G_\lambda(\xi, \xi)$ attains its maximum value with respect to ξ at $\xi = 1/2$:

$$G_\lambda(\xi, \xi) \leq \frac{1}{2\lambda^{1/2} \sinh \lambda^{1/2}} (\cosh \lambda^{1/2} - 1) = \frac{1}{2\lambda^{1/2}} \tanh \frac{\lambda^{1/2}}{2} = G_\lambda(1/2, 1/2).$$

As in Lemma 2.1

$$u(\xi)^2 \leq G_\lambda(\xi, \xi) (\|u\|^2 + \lambda \|u'\|^2) \leq G_\lambda(1/2, 1/2) (\|u\|^2 + \lambda \|u'\|^2),$$

where we have the equality at the point $1/2$ for $u(x) = \text{const } G_\lambda(x, 1/2)$. Hence,

$$\mathbb{V}(\xi, D) \leq \mathbb{V}(1/2, D),$$

and as in (2.30), (2.31) we obtain the parametric representation of $\mathbb{V}(D) := \mathbb{V}(1/2, D)$:

$$(3.95) \quad D(\lambda) = \frac{h(\lambda)}{g(\lambda)}, \quad V(D(\lambda)) = \frac{f(\lambda)^2}{g(\lambda)}, \quad \lambda \in [-\pi^2, \infty),$$

where

$$(3.96) \quad f(\lambda) = \frac{1}{2\lambda^{1/2}} \tanh \frac{\lambda^{1/2}}{2}, \quad g(\lambda) = -f'(\lambda), \quad h(\lambda) = f(\lambda) - \lambda g(\lambda),$$

and (3.93) is equivalent to the inequality

$$(3.97) \quad \mathbb{V}(D) \leq \sqrt{D}(1 - 2e^{-\sqrt{D}}) \quad \text{for } D \geq \pi^2.$$

Using the variational representation from Theorem 2.8 we have

$$(3.98) \quad \mathbb{V}(D) = \min_{\lambda \in (-\pi^2, \infty)} \frac{1}{2\sqrt{\lambda}} \tanh \frac{\lambda^{1/2}}{2} \cdot (\lambda + D).$$

To simplify notation we denote $d := \sqrt{D}$ and further set

$$(3.99) \quad \lambda_* = \lambda_*(D) = D(1 - 2\sqrt{D}e^{-\sqrt{D}})^2 = d^2(1 - 2de^{-d})^2.$$

As we shall see below $\lambda_*(D)$ contains the first terms of the asymptotic expansion as $D \rightarrow \infty$ of the unique solution of the first equation in (3.95), or, equivalently, of the unique point λ where the global minimum in (3.98) is attained. For the moment, however, we just substitute $\lambda_*(D)$

into the right-hand side of (3.98) and see that we prove (3.97) if we can show that the following inequality holds for all $d \geq \pi$:

$$(3.100) \quad \frac{1}{2} \frac{1}{d(1-2de^{-d})} \tanh \frac{d(1-2de^{-d})}{2} \left(d^2(1-2de^{-d})^2 + d^2 \right) \leq d(1-2e^{-d}),$$

or

$$(3.101) \quad \tanh \frac{d(1-2de^{-d})}{2} \left((1-2de^{-d})^2 + 1 \right) \leq 2(1-2e^{-d})(1-2de^{-d}).$$

Next, we use that $\tanh(u/2) \leq 1 - 2e^{-u} + 2e^{-2u}$, and we also observe that the quadratic polynomial $1 - 2x + 2x^2 = 2(x - 1/2)^2 + 1/2$ is monotone decreasing for $x \in [0, 1/2]$ so that

$$\tanh \frac{d(1-2de^{-d})}{2} \leq 1 - 2e^{-d(1-2de^{-d})} + 2e^{-2d(1-2de^{-d})} \leq 1 - 2e^{-d}(1+2d^2e^{-d}) + 2e^{-2d}(1+2d^2e^{-d})^2,$$

where we used that $e^{-d(1-2de^{-d})} \geq e^{-d}(1+2d^2e^{-d})$ which essentially is the inequality $e^x > 1+x$.

Combining the above we see that it suffices to establish for $d \geq \pi$ the inequality

$$(3.102) \quad F(d) := \left(1 - 2e^{-d}(1+2d^2e^{-d}) + 2e^{-2d}(1+2d^2e^{-d})^2 \right) \left((1-2de^{-d})^2 + 1 \right) - 2(1-2e^{-d})(1-2de^{-d}) \leq 0.$$

Simplifying $F(d)$ we obtain

$$F(d) = 4e^{-2d}F_1(d),$$

where

$$F_1(d) = -d^2 + 2d^2e^{-d} + 4d^3e^{-d} + 1 - 2de^{-d} + 2d^2e^{-2d} - 8d^3e^{-2d} + 8d^4e^{-3d} - 8d^5e^{-3d} + 8d^6e^{-4d}.$$

Next, dropping all negative terms except for $-d^2$ we have

$$F_1(d) < -d^2 + 1 + F_2(d), \quad F_2(d) := 2d^2e^{-d} + 4d^3e^{-d} + 2d^2e^{-2d} + 8d^4e^{-3d} + 8d^6e^{-4d}.$$

Each term in $F_2(d)$ is monotonely decreasing for $d \geq \pi$, and hence

$$F_1(d) < -\pi^2 + 1 + F_2(\pi) = -2.530 \dots < 0,$$

which proves (3.97).

To explain our choice of $\lambda_*(D)$ in (3.99) we find the asymptotics as $D \rightarrow \infty$ of the inverse function $\lambda = \lambda(D)$ since our $\lambda_*(D)$ contains the first two terms of this asymptotic expansion. We have

$$\begin{aligned} f(\lambda) &= \frac{1}{2\lambda^{1/2}} \tanh \frac{\lambda^{1/2}}{2} = \frac{1}{2\lambda^{1/2}} \frac{1 - e^{-\lambda^{1/2}}}{1 + e^{-\lambda^{1/2}}} = \frac{1}{2} \lambda^{-1/2} \left(1 - 2e^{-\lambda^{1/2}} + 2e^{-2\lambda^{1/2}} + \dots \right), \\ g(\lambda) &= \frac{1}{4} \lambda^{-3/2} \left(1 + (-2 - 2\lambda^{1/2})e^{-\lambda^{1/2}} + (2 + 4\lambda^{1/2})e^{-2\lambda^{1/2}} + \dots \right), \\ h(\lambda) &= \frac{1}{4} \lambda^{-1/2} \left(1 + (-2 + 2\lambda^{1/2})e^{-\lambda^{1/2}} + (2 - 4\lambda^{1/2})e^{-2\lambda^{1/2}} + \dots \right), \end{aligned}$$

and

$$\begin{aligned} D(\lambda) &= \frac{h(\lambda)}{g(\lambda)} = \frac{\lambda(1 + (-2 + 2\lambda^{1/2})e^{-\lambda^{1/2}} + (2 - 4\lambda^{1/2})e^{-2\lambda^{1/2}} + \dots)}{1 + (-2 - 2\lambda^{1/2})e^{-\lambda^{1/2}} + (2 + 4\lambda^{1/2})e^{-2\lambda^{1/2}} + \dots} = \\ &= \lambda + 4\lambda^{3/2}e^{-\lambda^{1/2}} + 8\lambda^2e^{-2\lambda^{1/2}} + \dots \end{aligned}$$

Hence the inverse function $\lambda(D)$ has the asymptotics

$$\lambda(D) = D(1 - 4\sqrt{D}e^{-\sqrt{D}} + \dots) = d^2(1 - 2de^{-d} + \dots)^2,$$

whose first two terms give (3.99).

Finally,

$$\begin{aligned}
\mathbb{V}(D) &= \frac{1}{2\sqrt{\lambda(D)}} \tanh \frac{\lambda(D)^{1/2}}{2} \cdot (\lambda(D) + D) = \\
&= \frac{1}{2} \frac{1}{d(1 - 2de^{-d} + \dots)} \tanh \frac{d(1 - 2de^{-d} + \dots)}{2} \left(d^2(1 - 2de^{-d} + \dots)^2 + d^2 \right) = \\
&= d \cdot \tanh \frac{d(1 - 2de^{-d} + \dots)}{2} \left(1 - 2de^{-d} + \dots \right) \left(1 + 2de^{-d} + \dots \right) = \\
&= d \cdot \tanh \frac{d(1 - 2de^{-d} + \dots)}{2} (1 + \dots) = d(1 - 2e^{-d} + \dots),
\end{aligned}$$

which proves sharpness and completes the proof. \square

Remark 3.20. Arguing similarly to Proposition 2.6 one can write down the three-term expansion of $\mathbb{V}(D)$ as $D \rightarrow \infty$:

$$(3.103) \quad \mathbb{V}(D) = D^{1/2} - 2D^{1/2}e^{-D^{-1/2}} - 2D^{1/2}(D-1)e^{-2D^{-1/2}} + o(e^{-3D^{-1/2}+\varepsilon}).$$

The third term is negative and hence inequality (3.97) holds for all sufficiently large D . However, as we have shown, (3.97) holds for *all* $D \geq \pi^2$.

3.3.4. *On a second order inequality on the interval.* We want to apply the above developed theory to the following one-dimensional interpolation inequality

$$(3.104) \quad \|u\|_\infty^2 \leq K \|u\|_{L_2(0,1)}^{3/2} \|u''\|_{L_2(0,1)}^{1/2}, \quad u \in H^2(0,1) \cap H_0^1(0,1)$$

Here, $Au(x) := u^{(4)}(x)$ with boundary conditions $u(0) = u(1) = u''(0) = u''(1) = 0$, $B = Id$, $\theta = 3/4$ and $\mathbb{A}(\lambda) = u^{(4)} + \lambda u$ and, in order to find the key function $G_\lambda(\xi, \xi)$, we need to solve the equation

$$u''''(x) + \lambda u(x) = \delta(x - \xi).$$

Using the orthonormal system of eigenfunctions $\{\sqrt{2} \sin \pi n x\}_{n=1}^\infty$ we obtain

$$(3.105) \quad G_\lambda(x, \xi) = 2 \sum_{n=1}^\infty \frac{\sin \pi n x \sin \pi n \xi}{\pi^4 n^4 + \lambda},$$

and setting $\lambda = 4a^4\pi^4$ to simplify the formulas below we have

$$G_\lambda(\xi, \xi) = \frac{2}{\pi^4} \sum_{n=1}^\infty \frac{\sin^2 \pi n \xi}{n^4 + 4a^4}.$$

Next, the identity $\sin^2 \pi n \xi = 1/2 - (e^{2\pi i n \xi} + e^{-2\pi i n \xi})/4$ and the Poisson summation formula

$$\sum_{n=-\infty}^\infty g(n+y) = \sqrt{2\pi} \sum_{n=-\infty}^\infty e^{2\pi i n y} \hat{g}(2\pi n)$$

give

$$\begin{aligned}
\sum_{n=1}^\infty \frac{\sin^2 \pi n \xi}{n^4 + 4a^4} &= \frac{1}{4} \left(\sum_{n=-\infty}^\infty \frac{1}{n^4 + 4a^4} - \sum_{n=-\infty}^\infty \frac{e^{2\pi i n \xi}}{n^4 + 4a^4} \right) = \\
&= \frac{1}{4} \frac{1}{(\sqrt{2}a)^3} \left(\sum_{n=-\infty}^\infty f(\sqrt{2}a2\pi n) - \sum_{n=-\infty}^\infty f(\sqrt{2}a2\pi(n + \xi)) \right) = \\
&= \frac{1}{4} \frac{1}{(\sqrt{2}a)^3} \frac{\pi\sqrt{2}}{2} \left(\sum_{n=-\infty}^\infty f_0(a2\pi n) - \sum_{n=-\infty}^\infty f_0(a2\pi(n + \xi)) \right),
\end{aligned}$$

where

$$f(y) := \int_{-\infty}^{\infty} \frac{e^{-ixy} dx}{x^4 + 1} = \frac{\pi\sqrt{2}}{2} f_0\left(\frac{|y|}{\sqrt{2}}\right), \quad f_0(x) = e^{-|x|}(\cos|x| + \sin|x|).$$

Combining the above, we obtain for the key function in Theorem 2.5

$$(3.106) \quad \lambda^{3/4} G_\lambda(\xi, \xi) = \frac{\sqrt{2}}{4} \left(\sum_{n=-\infty}^{\infty} f_0(a2\pi n) - \sum_{n=-\infty}^{\infty} f_0(a2\pi(n + \xi)) \right) =: \frac{\sqrt{2}}{4} S(a, \xi).$$

We now observe that the function $f_0(x)$ (which up to a constant factor is the fundamental solution of the operator (3.3.4) on the whole line) has positive local maximums at $x = 2\pi n$ (the one at 0 being the global) and negative local minimums at $x = \pi + 2\pi n$ (the ones at $\pm\pi$ being the global).

For a large enough and $\xi \leq 1/2$ the leading terms are

$$S(a, \xi) = f_0(0) - f_0(2\pi a\xi) + O(e^{-2\pi a}),$$

and since $f_0(x)$ has a negative global minimum at π , we see a “horn” of height $e^{-\pi}$ at $\xi = a^{-1}/2$, which gives the maximum value of $S(a, \xi)$ for large a ; see Fig. 4.

As for the global maximum of $S(a, \xi)$, we see that if $a = 1$ and $\xi = 1/2$, then the first and the second sums in (3.106) count one by one all the maximums and all the negative minimums of f_0 , respectively. Therefore

$$S(a, \xi) \leq S(1, 1/2) = \coth \frac{\pi}{2}.$$

Therefore we have proved the following result.

Theorem 3.21. *The sharp constant in inequality (3.104) is*

$$K = \frac{\sqrt{2}}{\sqrt[4]{27}} \cdot \coth \frac{\pi}{2} = \frac{\sqrt{2}}{\sqrt[4]{27}} \cdot 1.09033 \dots$$

The unique extremal function is given by (3.105) with $\lambda = 4\pi^4$ and $\xi = \frac{1}{2}$.

Remark 3.22. We point out that the sharp constant in (3.104) for $u \in H_0^2(0, 1)$ is the same as that on the whole line, namely, $\frac{\sqrt{2}}{\sqrt[4]{27}}$.

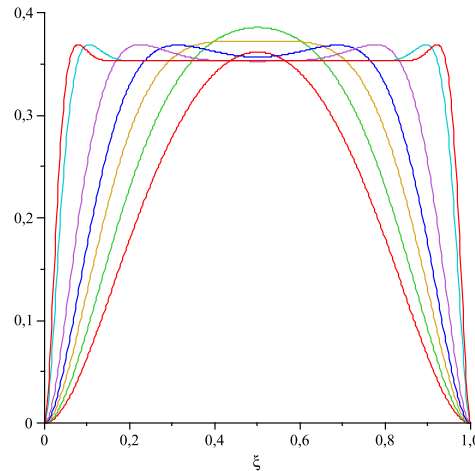


FIGURE 4. Plots of $\xi \rightarrow S(a, \xi)$ for $a = 2.5/\pi$ (red), $a = 1$ (green), $a = 4/\pi$ (brown), $a = 5/\pi$ (blue), $a = (7, 15, 20)/\pi$

3.3.5. *A second-order inequality in 3D.* In conclusion we consider a second-order inequality in a three dimensional domain $\Omega \subseteq \mathbb{R}^3$ for which the passage from $H_0^2(\Omega)$ to a wider space $H^2(\Omega) \cap H_0^1(\Omega)$ does not increase the constant in the corresponding interpolation inequality. This inequality was obtained in [27] by a somewhat problem specific method, nevertheless we present its proof in the framework of our general Theorem 2.5.

Theorem 3.23. *Let $\Omega \subseteq \mathbb{R}^3$ be an arbitrary domain. Let $\dot{H}_0^1(\Omega)$ be the completion of $C_0^\infty(\Omega)$ in the norm $\|\nabla \cdot\|_{L_2(\Omega)}$. Then for $u \in \dot{H}_0^1(\Omega) \cap \{u : \Delta u \in L_2(\Omega)\}$ the following inequality holds*

$$(3.107) \quad \|u\|_\infty^2 \leq \frac{1}{2\pi} \|\nabla u\| \|\Delta u\|,$$

where the constant is sharp and no extremal functions exist, unless $\Omega = \mathbb{R}^3$ and

$$u(x) = \mathcal{F}^{-1}(1/(|\xi|^2 + |\xi|^4)).$$

Proof. We first assume that Ω is a bounded domain with smooth boundary. Then by the elliptic regularity we have

$$u \in \dot{H}_0^1(\Omega) \cap \{u : \Delta u \in L_2(\Omega)\} = H^2(\Omega) \cap H_0^1(\Omega)$$

In accordance with Theorem 2.5 we have to consider the Green's function $G_\lambda(x, \xi)$ of the following 4th order elliptic equation

$$(3.108) \quad \begin{aligned} (\Delta^2 - \lambda \Delta) G_\lambda(x, \xi) &= -\Delta(-\Delta + \lambda) G_\lambda(x, \xi) = \delta(x - \xi), \\ G_\lambda(x, \xi) &= \Delta G_\lambda(x, \xi) = 0, \quad \xi \in \Omega, \quad x \in \partial\Omega. \end{aligned}$$

We denote by $g_D(x, \xi)$ the Green's function of the Dirichlet Laplacian:

$$-\Delta g_D(x, \xi) = \delta(x - \xi), \quad g_D(x, \xi) = 0, \quad \xi \in \Omega, \quad x \in \partial\Omega,$$

and by $g_H(x, \xi)$ the Green's function of the Helmholtz equation:

$$(-\Delta + \lambda) g_H(x, \xi) = \delta(x - \xi), \quad g_H(x, \xi) = 0, \quad \xi \in \Omega, \quad x \in \partial\Omega.$$

By the maximum principle we have

$$(3.109) \quad 0 < g_D(x, \xi) < \frac{1}{4\pi|x - \xi|}, \quad 0 < g_H(x, \xi) < \frac{e^{-\sqrt{\lambda}|x - \xi|}}{4\pi|x - \xi|},$$

where the functions on the right-hand sides are the corresponding fundamental solutions in \mathbb{R}^3 . Therefore

$$\begin{aligned} 0 < G_\lambda(x, \xi) &= \int_\Omega g_H(x, y) g_D(y, \xi) dy < \int_\Omega \frac{e^{-\sqrt{\lambda}|x - y|}}{4\pi|x - y|} \frac{1}{4\pi|y - \xi|} dy < \\ &< \int_{\mathbb{R}^3} \frac{e^{-\sqrt{\lambda}|x - y|}}{4\pi|x - y|} \frac{1}{4\pi|y - \xi|} dy = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{i(\xi - x) \cdot z} dz}{|z|^2(|z|^2 + \lambda)}, \end{aligned}$$

the function on the right-hand side being the fundamental solution of (3.108) in \mathbb{R}^3 , and, consequently,

$$G_\lambda(\xi, \xi) < \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{dz}{|z|^2(|z|^2 + \lambda)} = \frac{1}{4\pi\sqrt{\lambda}}.$$

Now (3.107) follows from Theorem 2.5. The constant $1/(2\pi)$ is clearly sharp, since we have the inequality with the same constant for $u \in H_0^2(\mathbb{R}^3)$, see (3.2), and (using extension by zero) for $u \in H_0^2(\Omega) \subset \dot{H}_0^1(\Omega) \cap \{u : \Delta u \in L_2(\Omega)\}$.

Exhausting an arbitrary $\Omega \subset \mathbb{R}^3$ with bounded domains Ω_n one can show that (3.107) holds in the general case (see [27] for the details). \square

Remark 3.24. Unlike the previous example, the fundamental solution of (3.108) in \mathbb{R}^3 is positive.

Acknowledgments. The authors would like to thank A.A.Laptev and S.I.Pokhozhaev for many helpful discussions.

This work was supported by the Russian Ministry of Education and Science (contract no. 8502). The work of A.A.I. was supported in part by the Russian Foundation for Fundamental Research, grants no. 12-01-00203 and no. 11-01-00339.

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